Armadale Academy
Higher Mathematics

Prelim
Revision Booklet

Contents

Straight Line

Perpendicular Gradients	Finding points of intersection	$m = tan\theta$	Finding the equation of a straight line	Equation of an altitude	Equation of a median	Points of intersection

1. Find the equation of the straight line which passes through the point (–2,6) and is :–
   (a) parallel to the line with equation $x = 3$.
   (b) perpendicular to the line with equation $y + 2x = 0$.
   (c) parallel to the line with equation $y - 3x = 4$.

2. Find the equation of the median PS of the triangle PQR where the coordinates of P, Q and R are :– (–2,3), (–3,–4) and (5,2) respectively.

3. Find the equation of the perpendicular bisector of the line joining M(2,–1) and N(8,3).

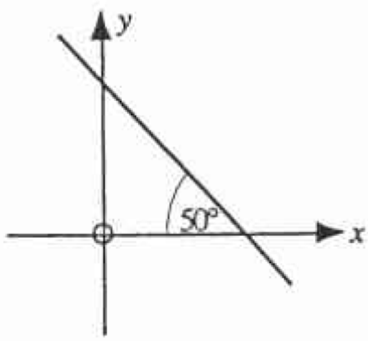
4. P(–3,5), Q(–1,–2) and R(5,1) are the vertices of a triangle PQR.
   Find the equation of PS, the altitude from P to QR.

5. K, L and M are the points (–5,–8), (12,–1) and (13,4) respectively.
   (a) Find the equation of KM.
   (b) If kite KLMN is completed, with KM the axis of symmetry, find the equation of LN and hence find the coordinates of the mid point of LN.

6. C has coordinates (4,7), D(–2,3) and E(1,9).
   (a) Find the equation of the line through the mid point of CD, parallel to DE.
   (b) Verify that this line passes through the mid point of CE.

7. Points G(0,–10), H(10,3) and I(–4,10) are vertices of ΔGHI. Find :–
   (a) the equations of the altitude GP and of median HQ of this triangle
   (b) the coordinates of the point of intersection of GP and HQ.

8. Find the gradient of this straight line. ⟶



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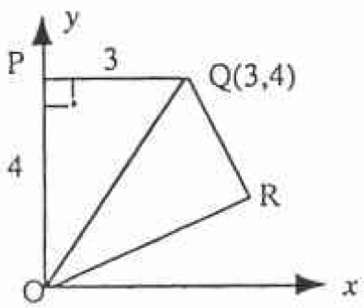


OPQR is a kite.
Calculate the gradient of line OR.
(correct to 2 decimal places)

Introduction	Fractions and Roots	More complex differentiation	Equations of tangents	Stationary points	Curve Sketching	Increasing /Decreasing functions	Optimisation	Graphs of Derivatives
								Applications of Differentiation

10. Differentiate the following with respect to x :–

(a) $f(x) = 5x^4 - 3x - 17$.

(b) $y = 6x^2 - \frac{3}{x}$.

(c) $y = \frac{9}{x^2} + x\sqrt{x}$.

(d) $f(x) = 3\sqrt{x}(x-3)$.

(e) $f(x) = \frac{6x^2 - 8x + 5}{2x}$.

11. If $f(x) = \sqrt[4]{x} - \frac{1}{\sqrt[4]{x}}$, find $f'(16)$.

12. A ball is thrown vertically upwards.
The height H metres of the ball s seconds after it is thrown, is given by the formula :–

$$H = 36s - 6s^2$$

(a) Find the rate of change of height, with respect to the time of the ball just as it is thrown.

(b) Find the speed of the ball after 3 seconds and explain your answer.

13. (a) Find the equation of the tangent to the curve with equation

$y = 5x^3 - 8x^2$ at the point where $x = 1$

(b) Find the x coordinate of each of the points on the curve

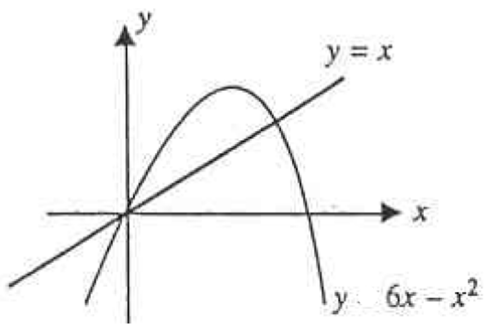
$$y = 2x^3 - 3x^2 - 12x + 12$$

at which the tangent is parallel to the x–axis.

14. Find the gradient of the tangent to the parabola

$y = 6x - x^2$ at (0,0)

Hence calculate the size of the angle between the line $y = x$ and this tangent.



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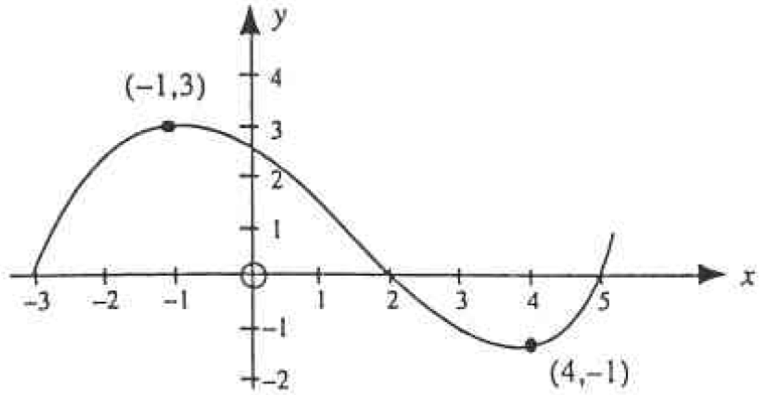


A sketch of a cubic function, $f(x)$, with domain $-3 \leq x \leq 5$, is shown.

Sketch the graph of the derived function $f'(x)$, for the same domain.

16. Find the maximum and minimum values of :–

$f(x) = 6x^2 - x^3$ in the closed interval $-2 \leq x \leq 2$.

17. Find the stationary values of the function defined by :–

$f(x) = 2 + 3x - x^3$

Hence, or otherwise, sketch the graph of $f(x)$ stating where the graph meets the axes.

18. (a) Given that $f(x) = \sqrt{x}(x - 3)$, where only the positive value of $\sqrt{x}$ is taken for each value of $x > 0$, find $f'(x)$.

(b) State the coordinates of the point on the curve $y = f(x)$ where $x = 4$ and obtain the the equation of the tangent to the curve at this point.

(c) Show that $(1,-2)$ is a stationary point on the curve and sketch the curve for values of x in the interval $0 \leq x \leq 4$ indicating the intersection with the x and y axes.

Domain and Range	Composite Functions	Inverse Functions	Exponential and Log Graphs	Graph Transformations

19. On a suitable set of real numbers, functions f and g are defined by :–

$$f(x) = \frac{1}{(x+3)} \text{ and } g(x) = \frac{1}{x} - 3$$

Find f(g(2)).

20. The functions h and k are defined by :–

$$h: x \longrightarrow x^2 \text{ and } k: x \longrightarrow x + 1$$

Find in its simplest form :– $h(k(p)) - k(h(p))$.

21. $g(x) = x^2 - 1$; $x \varepsilon R$, defines a function $g(x)$.

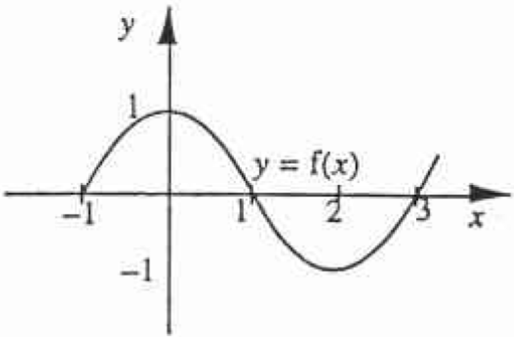
$h(x) = \frac{x-2}{x}$; $x \varepsilon R$, $x \neq 0$, defines a function $h(x)$.

(a) Explain why g has no inverse function.

(b) Define the inverse function $h^{-1}(x)$, stating a suitable domain.

22. The graph of $y = f(x)$ is shown.
On separate diagrams, sketch the graphs of :–

(a) $y = f(x) + 1$ (b) $y = f(x+1)$
(c) $y = -f(x) + 1$ (d) $y = -f(x+2)$
(e) $y = 2(f(x))$ (f) $y = 1 - f(x)$.



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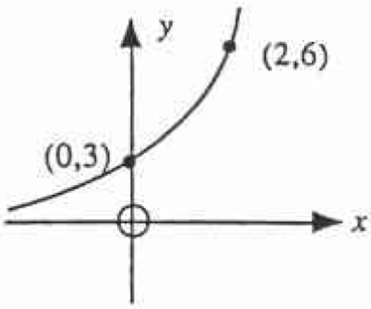
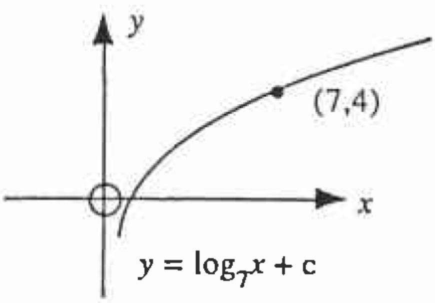


Part of the graph $y = a^x + b$ is shown.
Find the values of a and b.

24. The sketch shown opposite shows part of a logarithmic function.
Find the value of c.



25. Let p: $x \longrightarrow \sin x°$; q: $x \longrightarrow x^2$; r: $x \longrightarrow 1 - 2x$; be mappings on the set of Real Numbers R.

(a) Find a formula, in its simplest form, for the function s(x), such that s(x) = r(q(p(x))).
Hence, or otherwise, find the image of 30 under the mapping s.

(b) State, for each of p, q and r, whether there exists an inverse mapping on R and where it does exist, give the formula for it.

Completing the Square

Completing the Square

26. Express each of the following in the form $a(x + p)^2 + q$ by completing the square.

(a) $x^2 + 6x$ (b) $x^2 + 10x + 1$ (c) $5 + 4x - x^2$

(d) $2x^2 + 6x + 1$ (e) $4 + 6x - 3x^2$.

27. In each of the above cases, write down the maximum or minimum turning value, and the corresponding value of x each time.

28. (a) Express $2 + 4x - x^2$ in the form $a - (x + b)^2$.

(b) Hence write down the coordinates of the minimum turning point on the curve

$$f(x) = \frac{12}{2 + 4x - x^2}.$$

Radians	Exact Values	Solving Trig Equations

29. Sketch the graph of :–

(a) $y = 2\sin 3x°$ $(0 \le x \le 360)$

(b) $y = \cos(x - 60)° + 1$ $(0 \le x \le 360)$

30. Solve for $0 \le x \le 2\pi$, giving your answer to 2 decimal places when necessary :–

(a) $\cos^2 x = \frac{1}{4}$

(b) $12\cos^2 x - 5\cos x - 2 = 0$

(c) $2\sin x + \sqrt{3} = 0$

(d) $2\sin(2x + \frac{\pi}{6}) = 1.$

31. The minimum depth, d feet, of water in a marina, t hours after midnight, can be estimated by the function :–

$$d(t) = 21 + 12\cos(\frac{\pi}{6}t), \text{ where } 0 \le t \le 24.$$

(a) At midnight, a yacht, with a draft of 15 feet, is in the marina.(i.e. it needs a clear 15 ft depth of water to prevent being grounded).
By what time (24 hour clock) must it leave the marina to prevent being left aground ?

(b) What is the earliest time after that, the yacht can return to the marina ?

Polynomials

Introduction	Dividing Polynomials	Factorising Polynomials	Finding unknown coefficients	Solving Equations	Finding Functions from Graphs

51. A(1,0) and B(−2,0) are the two points at which the curve $y = x^4 + 2x^3 - 3x^2 - 4x + 4$ cuts the x–axis. By factorising the expression $x^4 + 2x^3 - 3x^2 - 4x + 4$ fully, prove that there are no other points of intersection with this axis.

52. Find the quotient and remainder when $6x^3 + 7x^2 - x - 2$ is divided by $2x - 1$.

53. Find n if $(x + 3)$ is a factor of $3x^3 + 2x^2 + nx + 6$, and factorise the expression fully when n has this value.

54. A function is defined $g(x) = x^3(3x + 2)$.
    (a) Find the stationary values of g and determine their natures.
    (b) Find where the graph of $g(x)$ cuts the x axis.
    (c) Sketch the graph of $g(x)$.

55. Show that the equation of the tangent to the curve $y = 5 - 2x^2 - x^3$ at $x = -2$ is $y + 4x + 3 = 0$.

56. A function is defined by $p(x) = x^3 + k$ where k is a constant.
    When $p(x)$ is divided by $x - 3$, the remainder is 36.
    Find k and hence solve the equation $p(-2x) = -18$.

57. (a) Find the stationary points of the function defined by $f(x) = 2 + x^2 - \frac{1}{3}x^3$ and determine their natures.
    (b) Show that the function has a value of zero for a replacement of x between $x = 3$ and $x = 4$ and find this value to one decimal place.
    (c) Sketch the graph of $f(x)$.

Quadratic Theory

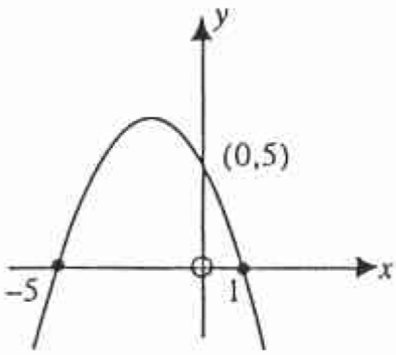
Quadratic Inequalities	The Discriminant	Using the Discriminant	Intersection of Parabola and Line

58. Find the values(s) of c for which the quadratic equation

$$x^2 - 2x + 21 = 2c(3x - 7)$$

has equal roots.

59. Find the equation of the parabola shown opposite.



60. Find t, given that $x^2 + (t - 3)x = -1$ has no real roots.

61. x is real and $m = \dfrac{x^2 + 4x + 10}{2x + 5}$.
By considering a quadratic equation in x, show that m cannot have a value between −3 and 2.

62. Given that $\dfrac{p}{x} + \dfrac{x+2}{p+1} = 2$, show that $p^2 + p(1 - 2x) + x^2 = 0$.
Hence determine the set of values for x for which p is real.

63. Find the condition for the quadratic equation $(mx + c)^2 = 8x$ to have equal roots.
Hence find the equation of the line through (0,4) which is tangent to the parabola that $y^2 = 8x$.

64. (a) Show that the line $y = 3x + c$ meets the parabola $y = 2x^2 + x - 4$ where

$$2x^2 - 2x + (-4 - c) = 0$$

(b) Find the value of c for the line to be a tangent to the parabola.

(c) Find the point of contact.

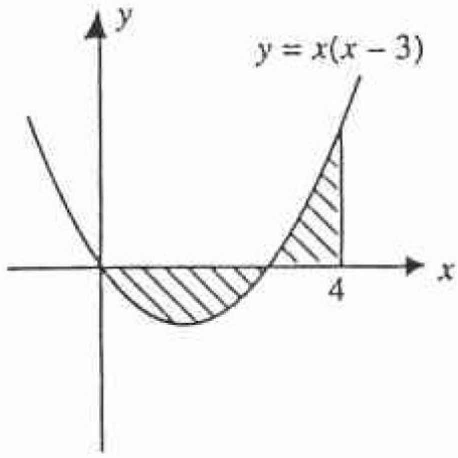
Introduction	Further Examples	Area under a curve	Area between curves	Differential Equations	Definite integrals

65. (a) $\int(\sqrt[4]{x}-3)\,dx$ (b) $\int \dfrac{6x^3-4x^2+2x}{x}\,dx$

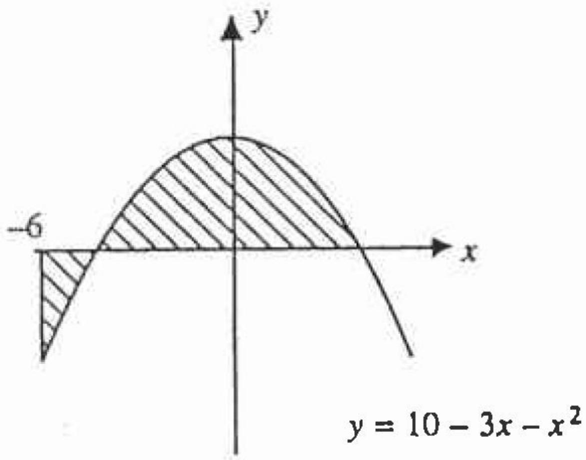
(c) $\int\left(x^2-\dfrac{1}{x^3}\right)dx$ (d) $\int_1^3\left(\dfrac{1+\sqrt{w}}{\sqrt{w}}\right)dw$

66. Calculate the shaded areas :–

(a)



(b)



67. The gradient of a tangent to a curve is given by $\dfrac{dy}{dx}=4-\dfrac{1}{x^2}$

If the curve passes through the point (1,6), find its equation.

68. Calculate the area enclosed between the functions

$g(x) = x$ and $h(x) = x^2 - 3x + 3$.

Sinx and Cosx	Further Calculus Sinx and Cosx	Chain Rule (Diff)	Integrating by inversing the chain rule

75. If $f(x) = \sqrt{1+x^2}$, find $f'(x)$.

[illegible]

$f'(x) = \frac{-1}{2}$ $\qquad 0 \le x < \pi.$

77. A curve has equation $y = (x+3)^{\frac{1}{2}}$.

Find the equation of the tangent at the point on the curve where $x = 6$.

[illegible]

79. A function of $f(x) = 1 + \cos(\pi/4 + x)$, where $0 \le x \le 2\pi$.

(a) Find the value of x for which $f(x) = 0$

(b) Find the values of x for which $f'(x) = 0$. Hence obtain the stationary values of f in the given interval.

80. F[illegible]sinx.

81. Evaluate :–

(a) $\int (6x-5)^5 \, dx$

(b) $\int_0^2 \sqrt{4x+1} \, dx$

(c) $\int_1^4 \frac{1}{(3+2x)^2} \, dx$

(d) $\int_0^{\frac{\pi}{3}}$ [illegible]

Equations of Lines

1. (a) $y = 3x + 8$ (b) $y = {}^{1}/_{2}x + 7$ (c) $y = 3x + 12$.

2. $3y + 4x = 1$.

3. $y - 1 = {}^{-3}/_{2}(x - 5)$.

4. $y - 5 = -2(x + 3)$.

5. (a) $3y - 2x = -14$ (b) $y + {}^{3}/_{2}x = 17$, (10,2).

6. (a) $y = 2x + 3$ (b) Proof.

7. (a) $y = 2x - 10$ and $y = {}^{1}/_{4}x + {}^{1}/_{2}$ (b) $({}^{14}/_{3}, {}^{-2}/_{3})$.

8. $-1{\cdot}2$.

9. $0{\cdot}29$.

Differentiation

10. (a) $20x^3 - 3$ (b) $12x + \frac{3}{x^2}$ (c) $\frac{-18}{x^3} + \frac{3}{2}x^{\frac{1}{2}}$

(d) $\frac{9}{2}(x^{\frac{1}{2}} - x^{-\frac{1}{2}})$ (e) $3 - \frac{5}{2x^2}$.

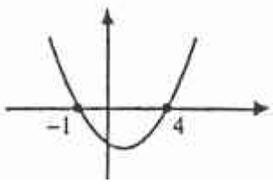
11. $\frac{5}{128}$.

12. (a) 36 m/s (b) 0 m/s; top of arc, stationary.

13. (a) $y + 3 = -(x - 1)$ (b) 2 and -1.

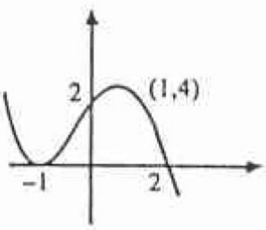
14. 6, $35{\cdot}5°$.

15.

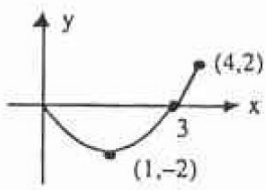


16. 32, 0.

17. 4 & 0



18. (a) $\frac{3}{2}\sqrt{x} - \frac{3}{2\sqrt{x}}$ (b) (4,2) $(y - 2) = \frac{9}{4}(x - 4)$ (c)



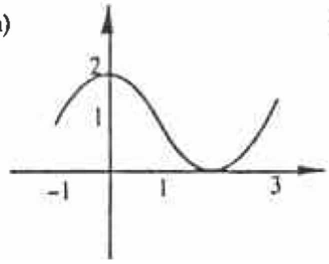
Graphs/Functions

19. 2.

20. 2p.

21. (a) Not in 1 to 1 correspondence (b) $\frac{2}{1-x}$ $x \in R;\ x \neq 1$

22. (a)



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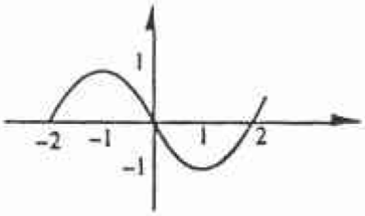

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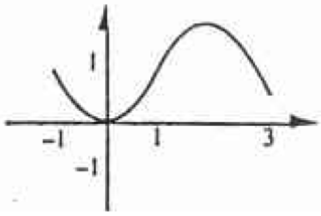

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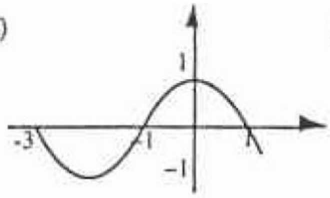

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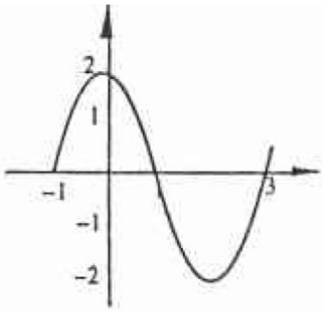
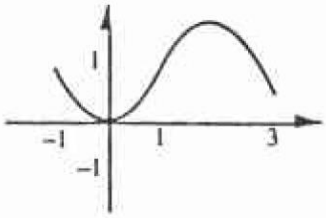


(f)



23. a = 2, b = 2.

24. c = 3.

25. (a) $1 - 2\sin^2 x°$; $^1/2$. (b) $r^{-1}(x) = \frac{1-x}{2}$; p, q – no inverses.

Completing the Square

26. (a) $(x+3)^2 - 9$ (b) $(x+5)^2 - 24$

(c) $9 - (x-2)^2$ (d) $2(x + {}^3/2)^2 - 3^1/2$

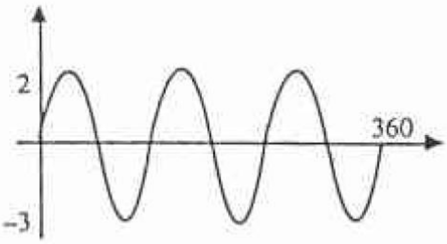
(e) $7 - 3(x-1)^2$.

27. (a) (–3,–9) (b) (–5,–24) (c) (2,9) (d) $(-^3/2, -3^1/2)$ (e) (1,7).

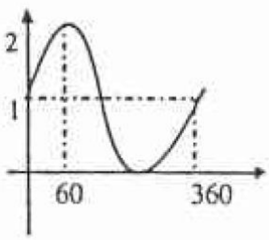
28. (a) $6 - (x-2)^2$ (b) minimum at (2,2).

Trig

29. (a)



(b)



30. (a) $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ (b) 0·84, 1·82, 4·46, 5·44 radians

(c) $\frac{4\pi}{3}, \frac{5\pi}{3}$ (d) $0, \frac{\pi}{3}, \pi, \frac{4\pi}{3}$.

31. (a) 0400 (b) 0800.

Polynomials

51. Proof showing 1 and −2 appear again as factors.

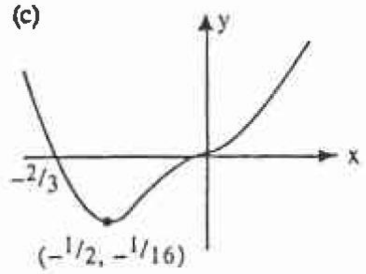
52. Quotient $3x^2 + 5x + 2$, remainder 0.

53. $n = -19$, $(x + 3)(3x - 1)(x - 2)$.

54. (a) (0,0) is a point of inflection $(-\frac{1}{2}, -\frac{1}{16})$ is a minimum turning point

(b) cuts x-axis (0,0) $(-\frac{2}{3},0)$

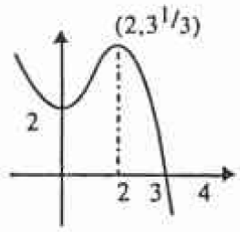
(c)



55. Proof.

56. $k = 9$, $x = \frac{3}{2}$.

57. (a) Min (0,2), Max $(2, 3\frac{1}{3})$ (b) Proof, 3·5. (c)



Quadratic Theory

58. $c = 2$ or $-\frac{10}{9}$.

59. $y = 5 - 4x - x^2$.

60. $(1 < t < 5)$.

61. Proof.

62. $(x \leq \frac{1}{4})$.

63. $(mc = 2)$, $y = \frac{1}{2}x + 4$.

64 (a) Proof (b) $c = \frac{-9}{2}$ (c) $(\frac{1}{2}, -3)$.

Integration

65. (a) $\frac{4}{5}x^{\frac{5}{4}} - 3x + c$ (b) $2x^3 - 2x^2 + 2x + c$

(c) $\frac{1}{3}x^3 + \frac{1}{2x^2} + c$ (d) $2\sqrt{3}$.

66. (a) $6^1/3$ units^2 (b) 61 units^2.

67. $y = 4x + \frac{1}{x} + 1$.

68. $1\frac{1}{3}$ units^2.

Further Calculus

75. $\frac{x}{\sqrt{1+x^2}}$.

76. $-\sin 2x$, $\frac{\pi}{12}, \frac{5\pi}{12}$

77. $y = \frac{1}{6}x + 2$.

78. $4\cos 2t - 3\cos^2 t \sin t$.

79. (a) $\frac{3\pi}{4}$ (b) $\frac{3\pi}{4}, \frac{7\pi}{4}$ 0 and 2.

80. Sketch showing $(\frac{\pi}{3}, \frac{3\sqrt{3}}{2})$ Max, $(\pi, 0)$ Inflection , $(\frac{5\pi}{3}, \frac{-3\sqrt{3}}{2})$ Min.

81. (a) $\frac{(6x-5)^6}{36} + c$ (b) $4\frac{1}{3}$ (c) $\frac{3}{55}$ (d) 3.