Essential Skills
Higher Maths

Content:

Original by G. Rennie - CDR edit Mar 22

FORMULAE LIST

Circle:

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

The equation $(x-a)^2 + (y-b)^2 = r^2$ represents a circle centre (a, b) and radius r.

Scalar Product : $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

or $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$, where $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Trigonometric formulae:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

Table of standard derivatives :

$f(x)$	$f'(x)$
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$

Table of standard integrals :

$f(x)$	$\int f(x)\,dx$
$\sin ax$	$-\frac{1}{a}\cos ax + C$
$\cos ax$	$\frac{1}{a}\sin ax + C$

Essential Skills 1.

The skills in this series of worksheets appear frequently.

These are the GIFTS you must take to succeed

Median of a Triangle (Non Calculator)

Find the equation of the Median from A in each:

1. A (4, 0), B (-1, -1) and C (11, 5)

2. A (-1, -5), B (-3, 5) and C (7, 3)

3. A (-2, -5), B (-1, 12) and C (7, -2)

4. A (4, 7), B (2, 1) and C (10, 1)

5. A (-1, 6), B (-2, -3) and C (6, 3)

6. A (-8, -3), B (2, 3) and C (6, -5)

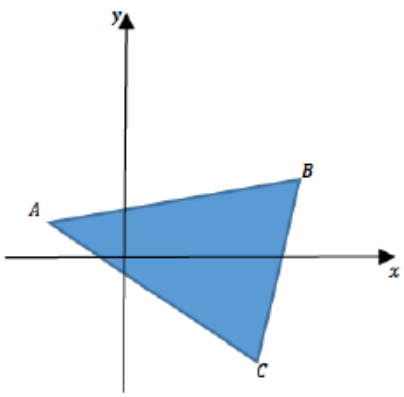
7. A (9, 8), B (3, 2) and C (11, 6)

8. A (5, 1), B (-2, 7) and C (6, -3)

9. A (3, -2), B (-1, -7) and C (7, 1)

10. A (5, 2), B (-1, 6) and C (-3, -2)

APPLYING QUESTION.



A triangle has vertices A (-3, 2), B (7, 4) and C (5, -6)

(a) Find the equations of medians AM and BN.

(b) Establish the coordinates of K, the point of intersection of AM and BN.

Essential Skills 2

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Equation of a Perpendicular Bisector (Non Calculator)

Find the equation of the perpendicular bisector of the line joining each pair of points:

1. A (4, 0) and B (2, 6)
2. A (-1, -4) and B (-3, 4)
3. A (-1, -3) and B (-7, -1)
4. A (4, 7) and B (10, 1)
5. A (2, -1) and B (8, 3)
6. A (-4, 3) and B (-2, 1)
7. A (9, 8) and B (3, 2)
8. A (-5, 4) and B (3, -2)
9. A (3, -2) and B (-7, -6)
10. A (-3, 2) and B (-3, 8)

APPLYING QUESTION

A is the point (-5, -1), B is (3, 3) and C is (4, -4) in triangle ABC

(a) Find the equation of the perpendicular bisector of the line AB

(b) Find the equation of the perpendicular bisector of the line AC

(c) Find the point of intersection of these perpendicular bisectors.

Essential Skills 3

The skills in this series of worksheets appear frequently.

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Altitude of a Triangle (Non Calculator)

Find the equation of the altitude from A in each:

1. A (2, 1), B (4, 7) and C (10, 1)

2. A (1, -5), B (-5, 5) and C (7, -7)

3. A (1, 14), B (0, -3) and C (9, 0)

4. A (-3, 1), B (4, 7) and C (8, -1)

5. A (-4, 4), B (-2, -6) and C (6, 2)

6. A (-1, 4), B (1, -4) and C (-7, 0)

7. A (1, 8), B (3, 14) and C (9, 8)

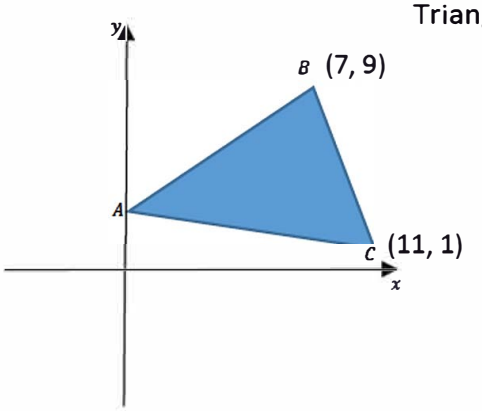
8. A (3, 9), B (-7, -1) and C (9, -9)

9. A (3, -2), B (-1, -5) and C (7, 1)

10. A (4, 8), B (-2, 4) and C (10, 4)

APPLYING QUESTION

Triangle ABC has vertex A on the y-axis.



(a) The equation of AB is $7y - 6x - 21 = 0$

State the coordinates of A.

(b) Find the equation of the altitude from A.

(c) Determine the co-ordinates of the point where the altitude from A meets the line BC.

Essential Skills 4

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Stationary Points

Find the co-ordinates and determine the nature of the stationary points:

1. $y = x^3 - 3x^2$

2. $f(x) = x^3 - 12x$

3. $f(x) = x^3 + 9x^2 + 24x - 18$

4. $y = 2x^3 - 7x^2 + 4x + 4$

5. $y = 2x^3 - 3x^2 - 36x + 17$

6. $f(x) = x^2(2x - 3)$

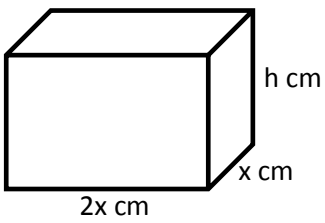
7. $f(x) = x^3 - 2x^2 - 4x + 1$

8. $y = (x - 1)(x - 2)^2$

9. $y = x(27 - x^2)$

10. $f(x) = 2x^2(2 - x^2)$

APPLYING QUESTIONS



1. An open top box measures x cm by $2x$ cm and has a depth of h cm. The outer surface has an area of 216cm².

(a) Show that the volume of the cuboid is given by $V(x) = 72x - \frac{2}{3}x^3$

(b) Find the value of x for which the volume is a maximum and calculate the volume.

2. A function f is defined by $f(x) = x(x^2 - 3)$, where $0 \leq x \leq 3$.

Find the maximum and minimum values of f.

Essential Skills 5

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Solving Quadratic Inequalities

By sketching the parabola, solve:

1. $x^2 - 4x \geq 0$
2. $x^2 + 14x + 33 \leq 0$
3. $x^2 - x - 20 > 0$
4. $x^2 - 9x + 8 < 0$
5. $x^2 - 16 \geq 0$
6. $3x^2 - 27 \leq 0$
7. $2x^2 + 5x - 3 < 0$
8. $7 - 6x - x^2 \geq 0$
9. $4x^2 \geq 8x + 5$
10. $6 + 7x \leq 3x^2$

APPLYING QUESTIONS

1. Find the values of x for which the function $f(x) = x^3 + 5x^2 - 8x + 3$ is increasing.

2. $x^2 - (k-2)x + 4 = 0$ has no real roots
   Find the range of values for k.

3. A circle has equation $x^2 + y^2 - 2kx - ky - 7k + 3 = 0$.
   Find the range of values for k.

Essential Skills 6

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Completing the Square

Write in the form $a(x+b)^2+c$:

1. $3x^2+6x+1$
2. $2x^2+12x-3$
3. $5x^2-10x-7$
4. $3x^2-18x+4$
5. $4x^2+24x+3$
6. $2x^2-20x-5$
7. $3-8x-x^2$
8. $5+16x-8x^2$
9. $2x^2-8x-2$
10. $3x^2+9x+1$

APPLYING QUESTION

(a) Write $2x^2-20x+54$ in the form $a(x+b)^2+c$

(b) Hence show that $y=\frac{2}{3}x^3-10x^2+54x-4$ is always increasing.

Essential Skills 7

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Equation of a Tangent to a Circle

Find the equation of each tangent at point P:

1. $(x-1)^2+(y-5)^2=25; P(-3,2)$
2. $x^2+y^2-6x-10y+16=0; P(6,2)$
3. $x^2+y^2-4x+6y+5=0; P(4,-1)$
4. $x^2+y^2=10; P(3,1)$
5. $(x+5)^2+y^2=40; P(-3,-6)$
6. $x^2+y^2+2y-24=0; P(4,2)$
7. $(x-3)^2+(y+2)^2=26; P(2,3)$
8. $x^2+y^2+2x+4y-3=0; P(-3,0)$
9. $(x+3)^2+(y-2)^2=4; P(-1,2)$
10. $x^2+y^2-8x+2y+1=0; P(4,3)$

APPLYING QUESTION

The circles with equations $x^2+y^2+14x+2y-50=0$ and $(x-5)^2+(y-8)^2=25$ touch at one common point.

(a) Find the coordinates of P, the point where the circles touch.

(b) Find the equation of the common tangent at P.

Essential Skills 8

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Intersection of Straight Line and a Circle

Find the coordinates of the points of intersection on each:

1. $x^2 + y^2 - 6x + 2y - 35 = 0$ and $y = 2x + 8$
2. $x^2 + y^2 - 6x - 4y + 8 = 0$ and $y = 2x + 1$
3. $x^2 + y^2 - 6x - 8y - 55 = 0$ and $x = 31 - 2y$
4. $x^2 + y^2 - 4x - 10y - 24 = 0$ and $y = 12 - x$
5. $x^2 + y^2 = 8$ and $y = 4 - x$
6. $x^2 + y^2 - 6x - 2y - 24 = 0$ and $y = x$
7. $x^2 + y^2 + 4x + 2y - 20 = 0$ and $y = 2x + 8$
8. $x^2 + y^2 + 18x + 20y + 81 = 0$ and $y = x + 1$
9. $x^2 + y^2 - 6x - 8y - 4 = 0$ and $y = 14 - x$
10. $x^2 + y^2 - 2x - 4y + 1 = 0$ and $x + y = 1$

APPLYING QUESTION

(a) Find the equation of a circle which has D (4, 1) and F (-2, -7) as its diameter. Leave your answer in the form $x^2 + y^2 + 2gx + 2fy + c = 0$.

(b) Establish the coordinates of the points of intersection between the circle and the line $y = x + 1$

Essential Skills 9

The skills in this series of worksheets appear frequently.

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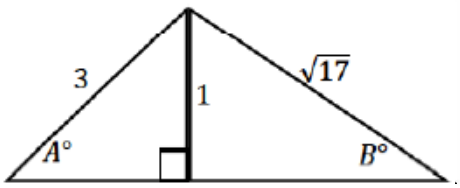
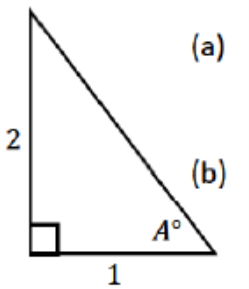
Trigonometric Formula

Calculate the exact value in each:

1. $sin75°$ given that $75° = 30° + 45°$
2. $cos15°$ given that $15° = 60° - 45°$
3. Given $tanx° = \frac{3}{4}$, find $sin2x°$
4. Given $tanx° = \frac{2}{3}$, find $cos2x°$
5. Given $tanA° = \frac{1}{2}$, find $\sin(A + 30)°$
6. Given $sinP° = \frac{12}{13}$, find $\cos(P + 30)°$
7. Given $cosB° = \frac{1}{\sqrt{10}}$, find $\sin(B - 45)°$
8. Given $tanx° = \frac{2}{5}$, find $sin2x°$
9. Given $tanA° = \frac{3}{4}$ and $tanB° = \frac{1}{2}$, find $\sin(A + B)°$
10. 

Show that $\cos(A - B)° = \frac{8\sqrt{2}+1}{3\sqrt{17}}$

APPLYING QUESTIONS

1. Given that $cos2x° = \frac{7}{25}$ find the value of $sinx°$. $(0 < x < 90°)$
2. 

(a) Given that $3A = 2A + A$ find the value of $sin3A°$

(b) Hence, or otherwise, find the value of $tan3A°$

Essential Skills 10

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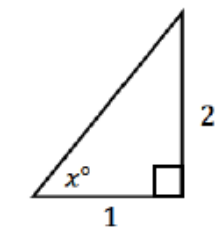
Related Angles (Non-Calculator)

Evaluate the exact value in each:

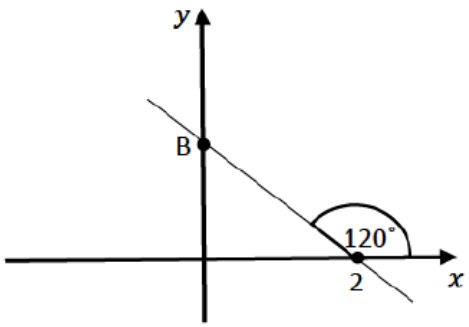
1. $sin150°$
2. $tan240°$
3. $cos315°$
4. $tan135°$
5. $sin210°$
6. $cos330°$
7. $cos225°$
8. $tan300°$
9. $sin240°$
10. $tan120° - sin330°$

APPLYING QUESTIONS

1. 

Find the exact value of $\cos(x - \frac{2\pi}{3})$

2. (a) Determine the equation of the line shown.



(b) Hence, write down the coordinates of B.

Essential Skills 11

The skills in this series of worksheets appear frequently.

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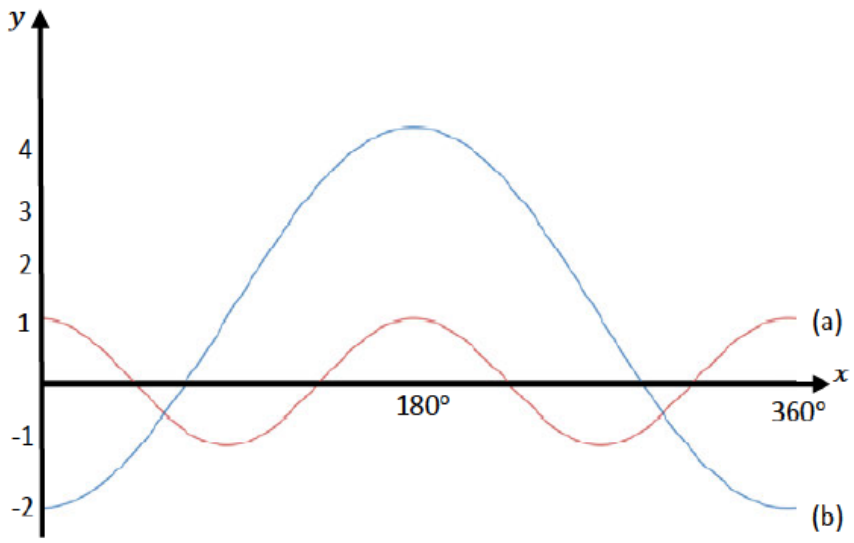
Trig Equations using Double Angle Formula

Solve each equation within the range shown:

1. $sin2x° - cosx° = 0$ $(0 \leq x \leq 360)$
2. $sin2x° + 3sinx° = 0$ $(0 \leq x \leq 360)$
3. $cos2x° + cosx° = 0$ $(0 \leq x \leq 360)$
4. $cos2x° - 4sinx° + 5 = 0$ $(0 \leq x \leq 360)$
5. $3cos2x° - cosx° + 1 = 0$ $(0 \leq x \leq 360)$
6. $2cos2x° + cosx° - 1 = 0$ $(0 \leq x \leq 360)$
7. $cos2x° + 3sinx° - 2 = 0$ $(0 \leq x \leq 2\pi)$
8. $5cos2x + 3sinx - 4 = 0$ $(0 \leq x \leq 2\pi)$
9. $cos2x = cosx$ $(0 \leq x \leq 2\pi)$
10. $2cos2x + 1 = 0$ $(0 \leq x \leq 2\pi)$

APPLYING QUESTION



(i) Find the equation of (a) in the form $y = cosbx°$.

(ii) Find the equation of (b) in the form $y = c - acosx°$.

(iii) Find algebraically the points of intersection of the graphs.

Essential Skills 12

The skills in this series of worksheets appear frequently.

These are the GIFTS you must take to succeed

Synthetic Division

1. Show that $(x-1)$ is a factor of x^3+4x^2-x-4 and factorise fully.
2. Show that $(x+2)$ is a factor of x^3+2x^2-4x-8 and factorise fully.
3. Show that $(x+1)$ is a factor of x^3-7x-6 and factorise fully.
4. Show that $(x-1)$ is a factor of $x^3-2x^2-11x+12$ and factorise fully.
5. Show that $(x+3)$ is a factor of $x^3+6x^2+11x+6$ and factorise fully.
6. Show that $(x-2)$ is a factor of $2x^3-3x^2-3x+2$ and factorise fully.
7. Show that $(x+1)$ is a factor of x^3-x^2-5x-3 and factorise fully.
8. Show that $x=-1$ is a root of $2x^3+7x^2+2x-3=0$ and find the other roots.
9. Show that $x=1$ is a root of $3x^3+x^2-3x-1=0$ and find the other roots.
10. Show that $x=2$ is a root of $x^3-x^2-8x+12=0$ and find the other roots.

APPLYING QUESTIONS

1. $(x-1)$ is a factor of $2x^3+px^2+2x-15$.
   Calculate p and factorise fully.

2. Find the coordinates of the points of intersection of $f(x)=x^3+4x^2-32x+30$ and $g(x)=5x-2x^2$

Essential Skills 13

The skills in this series of worksheets appear frequently.

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The Wave Function

Write in the required form in each, $k > 0, 0 \leq a \leq 360$:

1. $4cosx + 3sinx$ in the form $kcos(x - a)°$
2. $5sinx + 12cosx$ in the form $ksin(x + a)°$
3. $2cosx - 5sinx$ in the form $kcos(x + a)°$
4. $sinx - cosx$ in the form $ksin(x - a)°$
5. $\sqrt{2}cosx + 2sinx$ in the form $kcos(x - a)°$
6. $3sinx + \sqrt{5}cosx$ in the form $ksin(x + a)°$
7. $2cosx + sinx$ in the form $kcos(x + a)°$
8. $3sinx - 2cosx$ in the form $ksin(x + a)°$
9. $cosx + 3sinx$ in the form $ksin(x + a)°$
10. $6sinx + 8cosx$ in the form $kcos(x + a)°$

APPLYING QUESTIONS

1. (a) Write $2sinx + \sqrt{5}cosx$ in the form $ksin(x + a)°$ where $k > 0, 0 \leq a \leq 360$

   (b) State the minimum value of $y = 2sinx + \sqrt{5}cosx + 4$ and the value of x where it occurs.

2. (a) Express $4cosx - 3sinx$ in the form $kcos(x + a)$ where $k > 0, 0 \leq a \leq 2\pi$

   (b) Hence solve $4cosx - sinx = 2sinx - 3 \ \ (0 \leq x \leq 2\pi)$

Essential Skills 14

The skills in this series of worksheets appear frequently.

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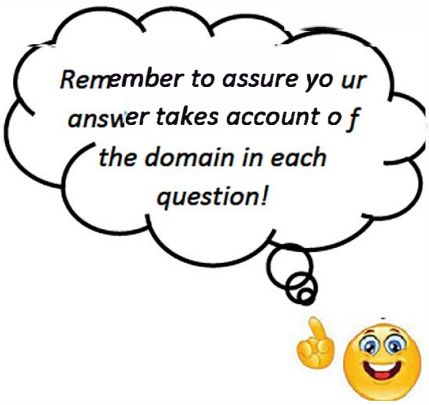
Logarithmic Equations

Solve for x in each:

1. $\log_a 6 + \log_a x = \log_a 12$
2. $\log_a 4x - \log_a 3 = \log_a 8$
3. $\log_a x + 2\log_a 4 = \log_a 80$
4. $\frac{1}{2}\log_2 x + \log_2 5 = \log_2 10$
5. $\log_a 81 - 3\log_a x = \log_a 3$
6. $\log_a(x+1) + \log_a(x-1) = \log_a 8$
7. $\log_a 4x + \log_a(x-1) = \log_a 3$
8. $\log_9(2x+5) - \log_9(x-5) = \log_9 \frac{x}{2}$
9. $\log_5(x+1) + \log_5(x-3) = 1$
10. $\log_7(x^2-1) - \log_7(x-1) = 2$



APPLYING QUESTIONS

1. Find the *x*-coordinate of the point where the graph of the curve with equation $y = log_3(x - 4) + 2$ intersects the *x*-axis.

2. Solve: $6\log_x 2 - 2\log_x 4 = 1$

Essential Skills 15

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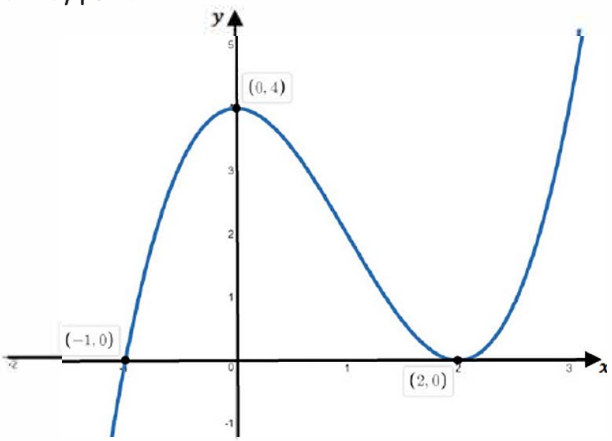
Related Graphs (Non Calculator)

1. The diagram below shows the graph of y = f(x). Sketch the graph of each the following on separate diagrams, indicating all key points:



(a) y = f(x) - 4

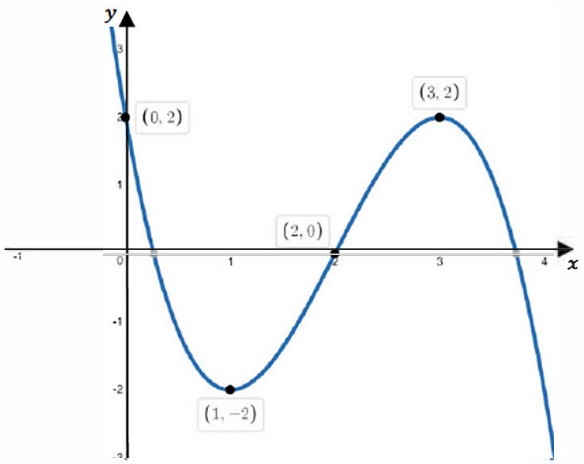
(b) y = f(x - 2)

(c) y = f(-x)

(d) y = 3 – f(x)

(e) y = f(2x)

2. The diagram below shows the graph of y = g(x). Sketch the graph of each the following on separate diagrams, indicating all key points:



(a) y = g(x + 3)

(b) y = g(x) + 3

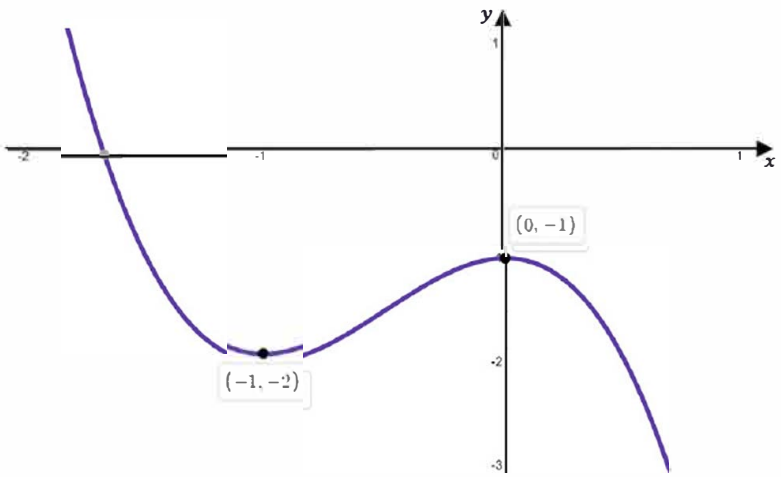
(c) y = g(-x) - 1

(d) y = -2g(x)

(e) y = g($\frac{1}{3}x$)

APPLYING QUESTION

The diagram below shows the graph of y = 3 – h(x + 1). Sketch the graph of h(x), indicating all key points.



Essential Skills 16

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Further Differentiation

Find the derivative of each, leaving your answers as positive indices:

1. $y = (x + 5)^4$
2. $f(x) = (2x - 1)^3$
3. $f(x) = (3x + 2)^3$
4. $y = (4x - 1)^{\frac{5}{4}}$
5. f(x) = $\frac{3}{(x+1)^3}$
6. $y = \sqrt{2x - 1}$
7. $y = (2x^2 + x)^3$
8. $f(x) = sin4x$
9. $y = -\cos(2x - \frac{\pi}{3})$
10. $y = 2cos^3x$

APPLYING QUESTIONS

1. If $f(x) = 2sin^2x$, show that $f'(x) = 2sin2x$ and hence calculate $f'\left(\frac{\pi}{3}\right)$.

2. A curve has equation $y = \frac{5}{4x+1}$, where $x \neq -\frac{1}{4}$

   Find the equation of the tangent to this curve at the point where x = 1.

Essential Skills 17

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Further Integration

Find the integral of each, leaving your answers as positive indices:

1. $\int 8(2x+1)^3\,dx$
2. $\int (x-1)^4\,dx$
3. $\int (3-2x)^3\,dx$
4. $\int (3x+1)^{\frac{1}{3}}\,dx$
5. $\int 2(4x+1)^{-2}\,dx$
6. $\int (9-x)^{-\frac{1}{2}}\,dx$
7. $\int \sqrt{3x-2}\,dx$

Essential Skills 18

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These are the GIFTS you must take to succeed

Differential Equations

Find the equations of the curves (y *or* $f(x)$) that satisfy each of the following conditions:

1. $\frac{dy}{dx} = 6x + 5$, passing (2, 21)
2. $\frac{dy}{dx} = 4x - 4$, passing (-1, 6)
3. $f'(x) = x^2$, where $f(3) = 13$
4. $f'(x) = 3x^2 - 6$, where $f(-1) = 8$
5. $\frac{dy}{dx} = 6x^2 + 8x + 5$, passing (-2, -12)
6. $f'(x) = 2(2 - 3x)$, where $f(1) = 1$
7. $\frac{dy}{dx} = \frac{9}{2}x^2 - 6x$, passing (2, 3)
8. $\frac{dy}{dx} = \frac{4}{x^3}$, passing (1, 1)
9. $\frac{dy}{dx} = 9(3x - 5)^2 + 5$, passing (2, 6)
10. $f'(x) = 6cos2x$, where $f\left(\frac{\pi}{12}\right) = \frac{5}{2}$

APPLYING QUESTIONS

1. The gradient of a tangent to a curve at each point (x , y) is given by $\frac{dy}{dx} = 3x(2x-1)$.

   If the curve passes through the point (–1, 10), find its equation.

2. The velocity of an object is given by $\frac{ds}{dt} = 9\sqrt{t} - 12$, where s is the distance in metres and t is the time in seconds.

   Find an expression for the displacement s, given that when $t = 0, s = 2$.

Essential Skills 19

The skills in this series of worksheets appear frequently.

These are the GIFTS you must take to succeed

Definite Integrals

Evaluate each of the following:

1. $\int_0^2 2x - 3\,dx$
2. $\int_0^3 3x^2\,dx$
3. $\int_1^4 \sqrt{x}\,dx$
4. $\int_1^3 x^2 - 2x\,dx$
5. $\int_{-1}^2 \frac{2}{x^3}dx$
6. $\int_0^2 3x^2 + 2x + 1\,dx$
7. $\int_1^4 \frac{x^2-1}{\sqrt{x}}dx$
8. $\int_1^2 (2x-1)^3\,dx$
9. $\int_{-1}^1 \frac{dx}{(x-2)^2}$

$y = x^2$

APPLYING QUESTIONS

1. Find p given that $\int_1^p 3x^2 - 5x\,dx = 6$

Essential Skills 20

The skills in this series of worksheets appear frequently.

These are the GIFTS you must take to succeed

Composite Functions

Find $f(g(x))$ and $g(f(x))$ for each of the following:

1. $f(x) = 8x + 3, g(x) = 1 - 2x$
2. $f(x) = x^2, g(x) = 1 + x$
3. $f(x) = 6x + 1, g(x) = 2x$
4. $f(x) = x^2 - 1, g(x) = 2x - 3$
5. $f(x) = x + 5, g(x) = \frac{1}{x}$
6. $f(x) = x + 1, g(x) = x^2 + x - 1$
7. $f(x) = \sqrt{x - 1}, g(x) = x^2 + 1$
8. $f(x) = 2x + 1, g(x) = \frac{1}{x-3}$
9. $f(x) = sinx, g(x) = 6x + 1$
10. $f(x) = cosx, g(x) = 2x^2 - 1$

APPLYING QUESTIONS

1. Given that $f(x) = \frac{1}{x^2-1}, \{x \neq \pm 1\}$ and $g(x) = x - 3$

   Find a formula for $h(x) = f(g(x))$, and state a suitable domain for $h(x)$.

2. Given that $f(x) = \frac{1}{1+x}, \{x \neq -1\}$, find $f\big(f(x)\big)$.

Essential Skills 21

The skills in this series of worksheets appear frequently.

These are the GIFTS you must take to succeed

Inverse Functions

Find $f^{-1}(x)$ for each of the following:

1. $f(x) = 6x + 1$
2. $f(x) = 6 - x$
3. $f(x) = \frac{1}{3}x - 2$
4. $f(x) = \frac{2}{5}x - 1$
5. $f(x) = \frac{x+5}{3}$
6. $f(x) = x^3 - 8,$
7. $f(x) = \sqrt{x-1}$ $\{x \geq 1\}$
8. $f(x) = 2x^3 + 1$
9. $f(x) = \frac{3}{x}$ $\{x \neq 0\}$
10. $f(x) = \frac{2}{3-x}$ $\{x \neq 3\}$

APPLYING QUESTIONS

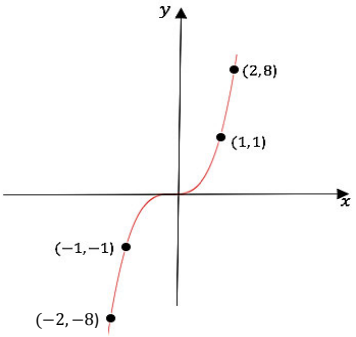
1. Given that $f(x) = \frac{x+1}{x-3}, \{x \neq 3\}$

   Find a formula for $f^{-1}(x)$, and state a suitable domain for $f^{-1}(x)$.

2. Explain why the function $f(x) = x^2 - 1, x \in \mathbb{R}$ does not have an inverse but that the restricted function $g(x) = x^2 - 1, x \geq 0, x \in \mathbb{R}$ does.

3. The graph of $f(x) = x^3$ is shown.

   Copy it and make a neat sketch of $f^{-1}(x)$ on the same diagram.



Essential Skills 22

The skills in this series of worksheets appear frequently.

These are the GIFTS you must take to succeed

Angle between a Line and the x-axis

Calculate the size of the angle of the line connecting each pair of points and the positive direction of the x-axis:

1. $A(3,2)\&\ B(7,6)$
2. $C(5,0)\&\ D(3,-4)$
3. $E(-1,3)\&\ F(3,5)$
4. $G(-2,6)\&\ H(4,0)$
5. $I(7,2)\&\ J(2,4)$

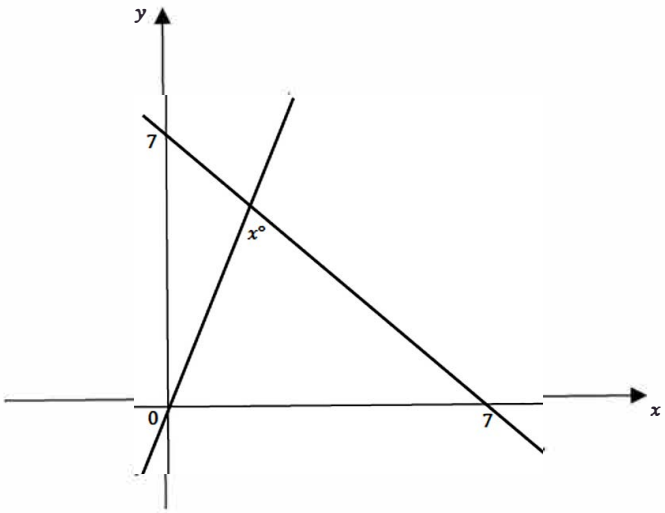
Calculate the gradient of the line given the angle it makes with the positive direction of the x-axis:

6. $50°$
7. $37 \cdot 5°$
8. $13°$
9. $123 \cdot 1°$
10. $116 \cdot 6°$

APPLYING QUESTION

The diagram shows the lines $y = 3x$ and $y = 7 - x$.

Calculate the size of angle $x°$



Essential Skills 23

The skills in this series of worksheets appear frequently.

These are the GIFTS you must take to succeed

Using the Natural Logarithm

Solve for x:

1. $3^x = 18$
2. $5^x = 90$
3. $12^x = 3000$
4. $4^{2x} = 35$
5. $2^{3x-1} = 11$
6. $0 \cdot 7^x = 0 \cdot 9$
7. $7^{2-3x} = 5$
8. $e^{0.6x} = 5 \cdot 2$
9. $e^{-0.3x} = 0 \cdot 16$
10. $50e^{-0.7x} = 45$

APPLYING QUESTIONS

1. Evaluate $\log_9 21$, giving your answer to 2 decimal places.

2. A radioactive element decays according to the formula $m_t = m_0 e^{-0.03t}$ where m_0 is the initial mass and t is the time in years.

   (a) What mass remains of the initial 200mg of the element after 40 years?

   (b) What is the half-life of this element?

3. A colony of ants is estimated to be growing according to the formula $P = 420e^{0.25t}$ where P is the population after t years.

   (a) What was the initial population of ants?

   (b) What is the population after 7 years?

   (c) How long will it take the population to increase by a factor of 10?

Essential Skills 24

The skills in this series of worksheets appear frequently.

These are the GIFTS you must take to succeed

Using Logarithms to Determine the Connection between Two Variables

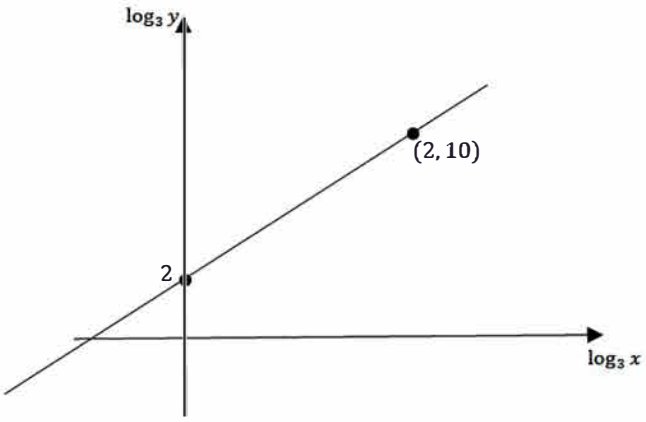
Obtain a formula for y in terms of x for each:

1. $\log_5 y = 3\log_5 x + \log_5 2$
2. $\log_2 y = 2\log_2 x + \log_2 0 \cdot 5$
3. $\log_3 y = \log_3 7 - \log_3 x$
4. $\log_{10} y = \log_{10} 13 - \frac{1}{2}\log_{10} x$
5. $\log_e y = 0 \cdot 2\log_e x + \log_e 3$
6. $\log_2 y = x\log_2 3 + \log_2 8$
7. $\log_5 y = x\log_5 0 \cdot 8 - \log_5 0 \cdot 2$
8. $\log_2 y = 4\log_2 x + 3$
9. $\log_9 y = 2\log_9 x + \frac{3}{2}$
10. $\log_6 y = x\log_6 \frac{1}{6} + 1$

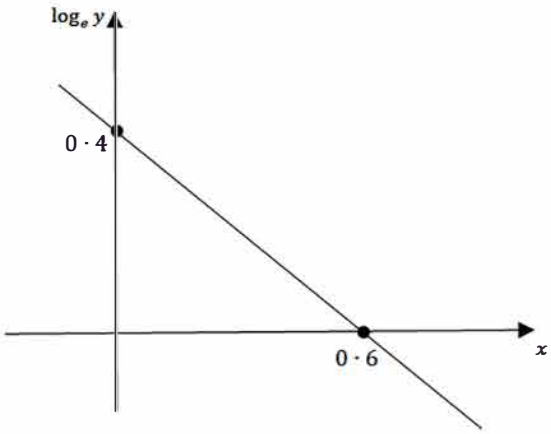
APPLYING QUESTIONS

Find a formula for each:

(a) $y = kx^n$



(b) $y = ab^x$



Essential Skills 25

The skills in this series of worksheets appear frequently.

These are the GIFTS you must take to succeed

Using the Discriminant

Find the value(s) of k given that each equation has equal roots:

1. $x^2 - 8x + k = 0$
2. $x^2 + kx + 16 = 0$
3. $kx^2 - 12x + 9 = 0$
4. $x^2 + 2kx + 9 = 0$
5. $x^2 + (k+1)x + 9 = 0$
6. $(k+1)x^2 - 2(k+3)x + 3k = 0$
7. $x^2 + (x+k)^2 - 8 = 0$
8. $x^2 + (kx-5)^2 = 9$
9. $kx^2 + (2k+1)x + k = 0$
10. $(7+2k)x^2 + kx + k = 0$

APPLYING QUESTIONS

1. The line $y = x + k$ is a tangent to the parabola $y = x^2 - 3x$.

   Find the value of k.

2. Given that $\frac{x^2+4x+10}{2x+5} = k$, form a quadratic equation in x and find the range of values of k for which it has 2 real and distinct roots.

3. Show that, if k is a real number, the roots of the equation $kx^2 + 3x - 3 = 2kx$ are always real.

Answers

Essential Skills 1	
1	$y - 2x + 8 = 0$
2	$y - 3x + 2 = 0$
3	$y - 2x + 1 = 0$
4	$y + 3x - 19 = 0$
5	$y + 2x - 4 = 0$
6	$6y - x + 10 = 0$
7	$y - 2x + 10 = 0$
8	$3y + x - 8 = 0$
9	$x = 3$
10	$y = 2$
AQ	(a) $3y + x - 3 = 0$ (b) $K(3, 0)$

Essential Skills 2	
1	$3y - x - 6 = 0$
2	$4y - x - 2 = 0$
3	$y - 3x - 10 = 0$
4	$y - x + 3 = 0$
5	$2y + 3x - 17 = 0$
6	$y - x - 5 = 0$
7	$y + x - 11 = 0$
8	$3y - 4x - 7 = 0$
9	$2y + 5x + 18 = 0$
10	$y = 6$
AQ	(a) $y + 2x + 1 = 0$ (b) $y - 3x + 1 = 0$ (c) (0, -1)

Essential Skills 3	
1	$y - x + 1 = 0$
2	$y - x + 6 = 0$
3	$y + 3x - 17 = 0$
4	$2y - x - 5 = 0$
5	$y + x = 0$
6	$y - 2x - 6 = 0$
7	$y - x - 7 = 0$
8	$y - 2x - 3 = 0$
9	$3y + 4x - 6 = 0$
10	$x = 4$
AQ	(a) A(0, 3) (b) $2y - x - 6 = 0$ (c) (8, 7)

Essential Skills 4	
1	maximum @ (0, 0); minimum @ (2, -4)
2	maximum @ (-2, 16); minimum @ (2, -16)
3	maximum @ (-4, -34); minimum @ (-2, -38)
4	maximum @ $(\frac{1}{3}, \frac{125}{27})$; minimum @ (2, 0)
5	maximum @ (-2, 61); minimum @ (3, -64)
6	maximum @ (0, 0); minimum @ (1, -1)
7	maximum @ $(-\frac{2}{3}, \frac{67}{27})$; minimum @ (2, -7)
8	maximum @ $(\frac{4}{3}, \frac{4}{27})$; minimum @ (2, 0)
9	minimum @ (-3, -54); maximum @ (3, 54)
10	maximums @ (-1, 2) & (1, 2); minimum @ (0, 0)
AQ	(1) (a) proof (b) $x = 6$; $V = 288cm^3$ (2) maximum 18 @ $x = 3$; minimum -2 @ $x = 1$

Answers

Essential Skills 5	
1	$x \leq 0, x \geq 4$
2	$-11 \leq x \leq -3$
3	$x < 4, x > 5$
4	$1 < x < 8$
5	$x \leq -4, x \geq 4$
6	$-3 \leq x \leq 3$
7	$-3 < x < \frac{1}{2}$
8	$-7 \leq x \leq 1$
9	$x \leq \frac{1}{2}, x \geq \frac{5}{2}$
10	$x \leq -\frac{2}{3}, x \geq 3$
AQ	1. $x < -4; x > \frac{2}{3}$ 2. $-2 < k < 6$ 3. $k < -6, k > \frac{2}{5}$

Essential Skills 6	
1	$3(x+1)^2 - 2$
2	$2(x+3)^2 - 21$
3	$5(x-1)^2 - 12$
4	$3(x-3)^2 - 23$
5	$4(x+3)^2 - 33$
6	$2(x-5)^2 - 55$
7	$19 - (x+4)^2$
8	$13 - 8(x-1)^2$
9	$2(x-2)^2 - 10$
10	$3(x+\frac{3}{2})^2 - \frac{23}{4}$
AQ	(a) $2(x-5)^2 + 4$ (b) $\frac{dy}{dx} > 0$ for all x, always increasing

Essential Skills 7	
1	$3y + 4x + 6 = 0$
2	$y - x + 4 = 0$
3	$y + x - 3 = 0$
4	$y + 3x - 10 = 0$
5	$3y - x + 15 = 0$
6	$3y + 4x - 22 = 0$
7	$5y - x - 13 = 0$
8	$y - x - 3 = 0$
9	$x = -1$
10	$y = 3$
AQ	(a) (1, 5) (b) $3y + 4x - 19 = 0$

Essential Skills 8	
1	(-3, 2)
2	(1, 3)
3	(7, 12)
4	(0, 12) & (9, 3)
5	(2, 2)
6	(-2, -2) & (6, 6)
7	(-6, -4) & (-2, 4)
8	(-3, -2) & (-17, -16)
9	(5, 9) & (8, 6)
10	(-1, 2) & (1, 0)
AQ	(a) $x^2 + y^2 - 2x + 6y - 15 = 0$ (b) (-4, -3) & (1, 2)

Answers

Essential Skills 9	
1	$\frac{1+\sqrt{3}}{2\sqrt{2}}$
2	$\frac{1+\sqrt{3}}{2\sqrt{2}}$
3	$\frac{24}{25}$
4	$\frac{5}{13}$
5	$\frac{2+\sqrt{3}}{2\sqrt{5}}$
6	$\frac{5\sqrt{3}-12}{26}$
7	$\frac{1}{\sqrt{5}}$
8	$\frac{20}{29}$
9	$\frac{2}{\sqrt{5}}$
10	Proof
AQ	(1) $\frac{3}{5}$ (2) (a) $-\frac{2}{5\sqrt{5}}$ (b) $\frac{2}{11}$

Essential Skills 10	
1	$\frac{1}{2}$
2	$\sqrt{3}$
3	$\frac{1}{\sqrt{2}}$
4	-1
5	$-\frac{1}{2}$
6	$\frac{\sqrt{3}}{2}$
7	$-\frac{1}{\sqrt{2}}$
8	$-\sqrt{3}$
9	$-\frac{\sqrt{3}}{2}$
10	$\frac{1-2\sqrt{3}}{2}$
AQ	(1) $\frac{2\sqrt{3}-1}{10}$ (2)(a) $y=-\sqrt{3}x+2\sqrt{3}$ (b) B$(0, 2\sqrt{3})$

Essential Skills 11	
1	$x=30°, 90°, 150°, 270°$
2	$x=0°, 180°, 360°$
3	$x=60°, 180°, 300°$
4	$x=90°$
5	$x=48.2°, 120°, 240°, 311.8°$
6	$x=41.4°, 180°, 318.6°$
7	$x=\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$
8	$x=\frac{\pi}{6}, \frac{5\pi}{6}, 3.34, 6.08$
9	$x=0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$
10	$x=\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
AQ	(i) $y=cos2x$ (ii) $y=1-3cosx$ (iii) $(60°, -\frac{1}{2})$ & $(300°, -\frac{1}{2})$

Essential Skills 12	
1	$(x-1)(x+1)(x+4)$
2	$(x+2)^2(x-2)$
3	$(x+1)(x-3)(x+2)$
4	$(x-1)(x-4)(x+3)$
5	$(x+3)(x+1)(x+2)$
6	$(x-2)(2x-1)(x+1)$
7	$(x+1)^2(x-3)$
8	$x=-1, x=\frac{1}{2}, x=-3$
9	$x=1, x=-\frac{1}{3}, x=-1$
10	$x=2, x=-3$
AQ	(1) $p=11$ (2) (1, 3), (-10, -250), (3, -3)

Answers

Essential Skills 13	
1	$5\cos(x-37)°$
2	$13\sin(x+67)°$
3	$\sqrt{14}\cos(x+68)°$
4	$\sqrt{2}\sin(x-45)°$
5	$\sqrt{6}\cos(x-55)°$
6	$\sqrt{14}\sin(x+37)°$
7	$\sqrt{5}\cos(x+333)°$
8	$\sqrt{13}\sin(x+326)°$
9	$\sqrt{10}\sin(x+18)°$
10	$10\cos(x+323)°$
AQ	(1) $3\sin(x+48)°$ min -1 @ 222° (2) (a) $5\cos(x+0.64)$ (b) $x=1.57, 3.43$

Essential Skills 14	
1	$x=2$
2	$x=6$
3	$x=5$
4	$x=4$
5	$x=3$
6	$x=3$
7	$x=\frac{3}{2}$
8	$x=10$
9	$x=4$
10	$x=48$
AQ	(1) $x=\frac{37}{9}$ (2) $x=4$

Essential Skills 15	
a	Correct shape; (-1, -4), (0, 0), (2, -4)
b	Correct shape; (1, 0), (2, 4), (4, 0)
c	Correct shape; (1, 0), (0, 4), (-2, 0)
d	Correct shape; (-1, 3), (0, -1), (2, 3)
e	Correct shape; $(-\frac{1}{2}, 0)$, (0, 4), (1, 0)
a	Correct shape; (-3, 2), (-2, -2), (0, 0), (1,2)
b	Correct shape; (0, 5), (1, 1), (2, 3), (3, 5)
c	Correct shape; (0, 1), (-1, -3), (-2, -1), (-3, 1)
d	Correct shape; (0, -4), (1, 4), (2, 0), (3, -4)
e	Correct shape; (0, 2), (3, -2), (6, 0), (9, 2)
AQ	Correct shape; (-2, 5) and (-1, 4)

Essential Skills 16	
1	$\frac{dy}{dx}=4(x+5)^3$
2	$f'(x)=6(2x-1)^2$
3	$f'(x)=9(3x+2)^2$
4	$\frac{dy}{dx}=5(4x-1)^{\frac{1}{4}}$
5	$f'(x)=-\frac{9}{(x+1)^4}$
6	$\frac{dy}{dx}=\frac{1}{\sqrt{2x-1}}$
7	$\frac{dy}{dx}=3(4x+1)(2x^2+x)^2$
8	$f'(x)=4cos4x$
9	$\frac{dy}{dx}=2\sin(2x-\frac{\pi}{3})$
10	$\frac{dy}{dx}=-6sinxcos^2x$
AQ	(1) Proof, $\sqrt{3}$ (2) $5y+4x-9=0$

Answers

Essential Skills 17	
1	$(2x+1)^4+c$
2	$\frac{1}{5}(x-1)^5+c$
3	$-\frac{1}{8}(3-2x)^4+c$
4	$\frac{1}{4}(3x+1)^{\frac{4}{3}}+c$
5	$-\frac{1}{2(4x+1)}+c$
6	$-2\sqrt{9-x}+c$
7	$\frac{2}{9}(3x-2)^{\frac{3}{2}}+c$

Essential Skills 18	
1	$y=3x^2+5x-1$
2	$y=2x^2-4x$
3	$f(x)=\frac{1}{3}x^3+4$
4	$f(x)=x^3-6x+5$
5	$y=2x^3+4x^2+5x-2$
6	$f(x)=4x-3x^2$
7	$y=\frac{3}{2}x^3-3x^2+3$
8	$y=3-\frac{2}{x^2}$
9	$y=(3x-5)^3+5$
10	$f(x)=3sin2x+1$
AQ	(1) $y=2x^3-\frac{3}{2}x^2+\frac{27}{2}$ (2) $s=6\sqrt{t^3}-12t+2$

Essential Skills 19	
1	-2
2	27
3	$\frac{14}{3}$
4	$\frac{2}{3}$
5	$\frac{3}{4}$
6	14
7	$\frac{8}{3}$
8	$\frac{63}{8}$
9	$\frac{4}{3}$
10	$\frac{1-\sqrt{3}}{4}$
AQ	(1) p = 3

Essential Skills 20	
1	$f(g(x))=11-16x \quad g(f(x))=-16x-5$
2	$f(g(x))=x^2+2x+1 \quad g(f(x))=1+x^2$
3	$f(g(x))=12x+1 \quad g(f(x))=12x+2$
4	$f(g(x))=4x^2-12x+8 \quad g(f(x))=2x^2-5$
5	$f(g(x))=\frac{1+5x}{5} \quad g(f(x))=\frac{1}{x+5}$
6	$f(g(x))=x^2+x \quad g(f(x))=x^2+3x+1$
7	$f(g(x))=x \quad g(f(x))=x$
8	$f(g(x))=\frac{x-1}{x-3} \quad g(f(x))=\frac{1}{2x-2}$
9	$f(g(x))=\sin(6x-1) \quad g(f(x))=6sinx-1$
10	$f(g(x))=\cos(2x^2-1) \quad g(f(x))=cos2x$
AQ	(1) $h(x)=\frac{1}{x^2-6x+8}, x\neq 2,4$ (2) $f(f(x))=\frac{1+x}{2+x}$

Answers

Essential Skills 21	
1	$f^{-1}(x) = \frac{x-1}{6}$
2	$f^{-1}(x) = 6 - x$
3	$f^{-1}(x) = 3(x+2)$
4	$f^{-1}(x) = \frac{5(x+1)}{2}$
5	$f^{-1}(x) = 3x - 5$
6	$f^{-1}(x) = \sqrt[3]{x+8}$
7	$f^{-1}(x) = x^2 + 1$
8	$f^{-1}(x) = \sqrt[3]{\frac{x-1}{2}}$
9	$f^{-1}(x) = \frac{3}{x}$
10	$f^{-1}(x) = \frac{3x-2}{x}$
AQ	(1) $f^{-1}(x) = \frac{1+3x}{x-1}$, $x \neq 1$ (2) Inverse needs the domain restriction to work (3) Suitable curve reflected in $y = x$

Essential Skills 22	
1	45°
2	63.4°
3	26.6°
4	135°
5	158.2°
6	1.19
7	0.77
8	0.23
9	-1.53
10	-2
AQ	63.4°

Essential Skills 23	
1	$x = 2.63$
2	$x = 2.80$
3	$x = 3.22$
4	$x = 1.28$
5	$x = 1.49$
6	$x = 0.30$
7	$x = 0.39$
8	$x = 2.75$
9	$x = 6.11$
10	$x = 0.15$
AQ	1. $x = 1.39$ 2. (a) 60 2kg (b) 23 years 3. (a) 420 (b) 2416 ants (c) 9.2 years

Essential Skills 24	
1	$y = 2x^3$
2	$y = 0.5x^2$
3	$y = \frac{7}{x}$
4	$y = \frac{13}{\sqrt{x}}$
5	$y = 3x^{0.2}$
6	$y = 8 \times 3^x$
7	$y = \frac{0.8^x}{0.2}$
8	$y = 8x^4$
9	$y = 27x^2$
10	$y = 6 \times \frac{1^x}{6}$
AQ	(a) $y = 9x^4$ (b) $y = 1.49 \times 0.51^x$

Answers

Essential Skills 25	
1	$k = 16$
2	$k = 8, k = -8$
3	$k = 4$
4	$k = 3, k = -3$
5	$k = -7, k = 5$
6	$k = -\frac{3}{2}, k = 3$
7	$k = 4, k = -4$
8	$k = -\frac{4}{3}, k = \frac{4}{3}$
9	$k = -\frac{1}{4}$
10	$k = -4, k = 0$
AQ	(1) $k = 16$ (2) $k < -3, k > 2$ (3) Min value 9 so $b^2 - 4ac > 0$ for all k