Armadale Academy

Higher Prelim Revision

FORMULAE LIST

Circle:

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

The equation $(x-a)^2 + (y-b)^2 = r^2$ represents a circle centre (a, b) and radius r.

Scalar Product: $\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between **a** and **b**

or $\mathbf{a.b} = a_1b_1 + a_2b_2 + a_3b_3$ where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Trigonometric formulae:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$= 2\cos^2 A - 1$$

$$= 1 - 2\sin^2 A$$

Table of standard derivatives:

$f(x)$	$f'(x)$
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$

Table of standard integrals:

$f(x)$	$\int f(x)dx$
$\sin ax$	$-\frac{1}{a}\cos ax + c$
$\cos ax$	$\frac{1}{a}\sin ax + c$

HIGHER MATHEMATICS

Mixed Revision 1 (Non-Calculator)

1. (Grade C)

Find the equation of the perpendicular bisector of the line joining A(2, −1) and B(8, 3).

2. (Grade C)

Find the value of $\int_1^4 \sqrt{x}\, dx$.

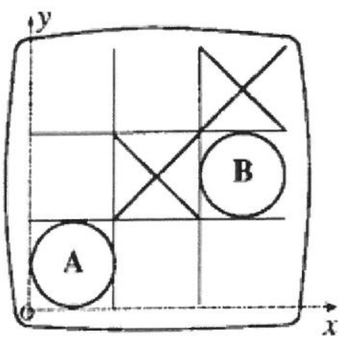
3. (Grade C)

For acute angles P and Q, $\sin P = \frac{12}{13}$ and $\sin Q = \frac{3}{5}$.

Show that the exact value of $\sin(P + Q)$ is $\frac{63}{65}$.

4. (Grade C)

This diagram shows a computer-generated display of a game of noughts and crosses.
Relative to the coordinate axes which have been added to the display, the "nought" at A is represented by a circle with equation $(x-2)^2 + (y-2)^2 = 4$.



(a) Find the centre of the circle at B.
(b) Find the equation of the circle at B.

5. (Grade C)

Find k if $x - 2$ is a factor of $x^3 + kx^2 - 4x - 12$.

6. (Grade C)

Solve algebraically the equation $\sin 2x^\circ + \sin x^\circ = 0, \quad 0 \le x < 360$.

13. (Grade A/B)

Given that k is a real number, show that the roots of the equation $kx^2+3x+3=k$ are always real numbers.

ANSWERS (Mixed Revision 1)

1. $y-1=-\frac{3}{2}(x-5)$ *or* $3x+2y=17$

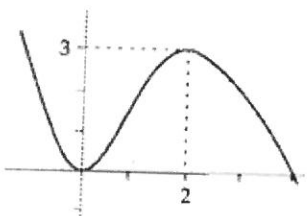
2. $\int_1^4 \sqrt{x}dx=\frac{14}{3}$

3. $\sin(P+Q)=\sin P\cos Q+\cos P\sin Q=\left(\frac{12}{13}\times\frac{4}{5}\right)+\left(\frac{5}{13}\times\frac{3}{5}\right)=\frac{63}{65}$

4.(a) $(10, 6)$ (b) $(x-10)^2+(y-6)^2=4$

5. $k=3$ **6.** $x=0, 120, 180, 240$ **7.** $\int_{-3}^{0}(2x+3)^2\,dx=9$

8. [*Hint*: write $y=-f(x)+2$] **9.** $f'\left(\frac{\pi}{6}\right)=\frac{3\sqrt{3}}{2}$



10. The equation represents a circle when $g^2+f^2-c>0 \Rightarrow 13>c$ $(or\ c<13)$

11. $a=3,\ b=2$

12.(a) $7-2x-x^2=8-(x+1)^2$ (b) max value = 8

13. $kx^2+3x+(3-k)=0 \Rightarrow b^2-4ac=(2k-3)^2\geq 0$ for all k,
hence the roots are always real

HIGHER MATHEMATICS

Mixed Revision 2 (Calculator)

1. (Grade C)

(*a*) Given that $x + 2$ is a factor of $2x^3 + x^2 + kx + 2$, find the value of k.

(*b*) Hence solve the equation $2x^3 + x^2 + kx + 2 = 0$ when k takes this value.

2. (Grade C)

A curve has equation $y = x - \frac{16}{\sqrt{x}}, \quad x > 0$.

Find the equation of the tangent at the point where $x = 4$.

3. (Grade C)

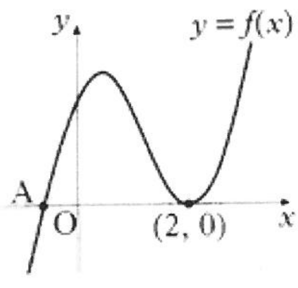
Find $\int \frac{(x^2 - 2)(x^2 + 2)}{x^2} dx, \quad x \neq 0$

4. (Grade C)

Given that $f(x) = (5x - 4)^{\frac{1}{2}}$, evaluate $f'(4)$.

5. (Grade C)

The diagram shows part of the graph of the curve with equation $y = 2x^3 - 7x^2 + 4x + 4$.



(*a*) Find the x-coordinate of the maximum turning point.

(*b*) Factorise $2x^3 - 7x^2 + 4x + 4$.

(*c*) State the coordinates of the point A and hence find the values of x for which $2x^3 - 7x^2 + 4x + 4 < 0$.

9. (Grade A/B)

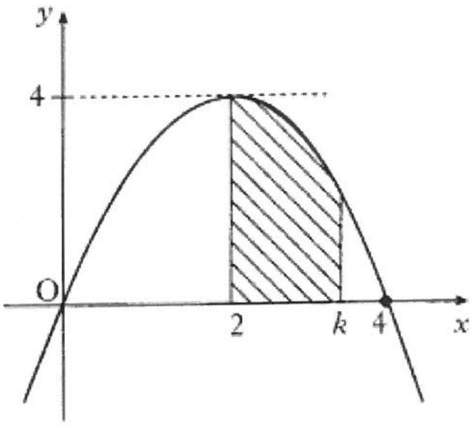
The parabola shown crosses the x-axis at (0, 0) and (4, 0), and has a maximum at (2, 4).

The shaded area is bounded by the parabola, the x-axis and the lines $x = 2$ and $x = k$.

(*a*) Find the equation of the parabola.

(*b*) Hence show that the shaded area, A, is given by

$$A = -\frac{1}{3}k^3 + 2k^2 - \frac{16}{3}.$$



10. (Grade A/B)

Solve the equation $3\cos 2x° + \cos x° = -1$ in the interval $0 \leq x \leq 360$.

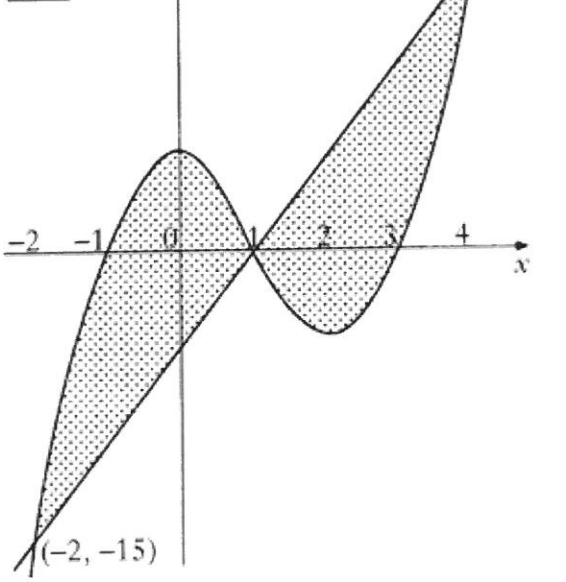
11. (Grade A/B)

A firm asked for a logo to be designed involving the letters A and U. Their initial sketch is shown in the hexagon.

A mathematical representation of the final logo is shown in the coordinate diagram.

The curve has equation $y = (x + 1)(x - 1)(x - 3)$ and the straight line has equation $y = 5x - 5$. The point (1, 0) is the centre of half-turn symmetry.

Calculate the total shaded area.



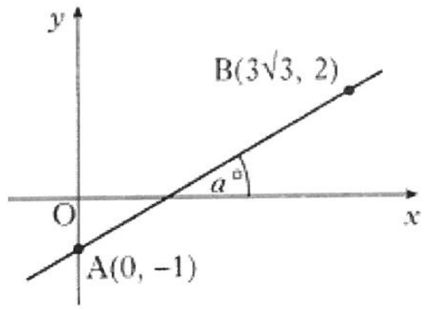
Mixed Revision 3 (Non-Calculator)

1. (Grade C)

The point P(2, 3) lies on the circle $(x + 1)^2 + (y - 1)^2 = 13$. Find the equation of the tangent at P.

2. (Grade C)

Find the size of the angle $a°$ that the line joining the points A(0, −1) and B($3\sqrt{3}$, 2) makes with the positive direction of the x-axis.



3. (Grade C)

A recurrence relation is defined by $u_{n+1} = pu_n + q$, where $-1 < p < 1$ and $u_0 = 12$.

(*a*) If $u_1 = 15$ and $u_2 = 16$, find the values of p and q.

(*b*) Find the limit of this recurrence relation as $n \to \infty$.

4. (Grade C)

Given that $f(x) = \sqrt{x} + \frac{2}{x^2}$, find $f'(4)$.

5. (Grade C)

Functions $f(x) = \frac{1}{x - 4}$ and $g(x) = 2x + 3$ are defined on suitable domains.

(*a*) Find an expression for $h(x)$ where $h(x) = f(g(x))$.

(*b*) Write down any restriction on the domain of h.

10. (Grade A/B)

Two sequences are generated by the recurrence relations $u_{n+1} = au_n + 10$ and $v_{n+1} = a^2v_n + 16$.

The two sequences approach the same limit as $n \to \infty$.

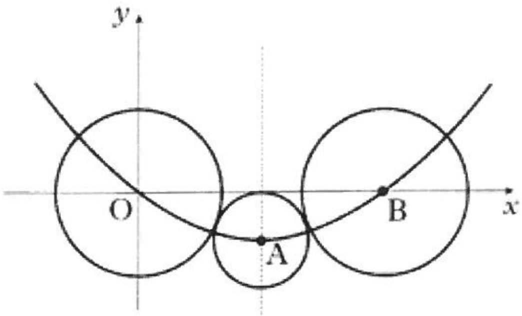
Determine the value of a and evaluate the limit.

11. (Grade A/B)

Show that the line with equation $y = 2x + 1$ does not intersect the parabola with equation $y = x^2 + 3x + 4$.

12. (Grade A/B)

- O, A and B are the centres of the three circles shown in the diagram below.
- The two outer circles are congruent and each touches the smallest circle.
- Circle centre A has equation $(x - 12)^2 + (y + 5)^2 = 25$.
- The three centres lie on a parabola whose axis of symmetry is shown by the broken line through A.



(*a*) (i) State the coordinates of A and find the length of the line OA.

(ii) Hence find the equation of the circle with centre B.

(*b*) The equation of the parabola can be written in the form $y = px(x + q)$.

Find the values of p and q.

ANSWERS (Mixed Revision 3)

Mixed Revision 4 (Calculator)

1. (Grade C)

$f(x) = 6x^3 - 5x^2 - 17x + 6$.

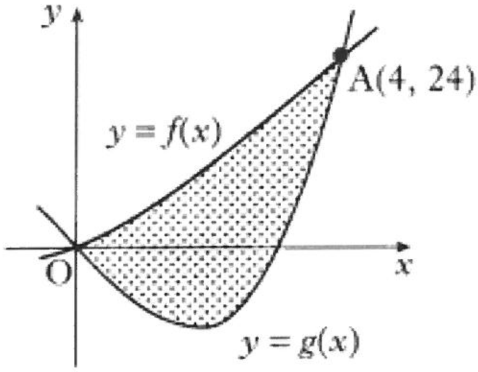
(*a*) Show that $(x - 2)$ is a factor of $f(x)$.

(*b*) Express $f(x)$ in its fully factorised form.

2. (Grade C)

The incomplete graphs of $f(x) = x^2 + 2x$ and $g(x) = x^3 - x^2 - 6x$ are shown in the diagram. The graphs intersect at A(4, 24) and the origin.

Find the shaded area enclosed between the curves.



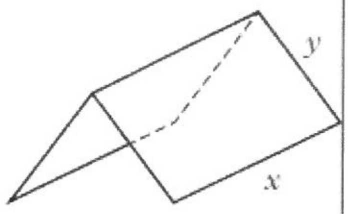
4. (Grade A/B)

A manufacturer is asked to design an open-ended shelter, as shown, subject to the following conditions.

Condition 1

The frame of a shelter is to be made of rods of two different lengths:

- x metres for top and bottom edges;
- y metres for each sloping edge.



Condition 2

The frame is to be covered by a rectangular sheet of material.

The total area of the sheet is $24\,\text{m}^2$.

(*a*) Show that the total length, L metres, of the rods used in a shelter is given by

$$L = 3x + \frac{48}{x}.$$

(*b*) These rods cost £8·25 per metre.

To minimise production costs, the total length of rods used for a frame should be as small as possible.

(i) Find the value of x for which L is a minimum.

(ii) Calculate the minimum cost of a frame.

ANSWERS (Mixed Revision 4)

1.(b) $f(x) = (x-2)(3x-1)(2x+3)$

2. Shaded area $= \dfrac{128}{3}$ units^2

3.(a) P(−3, −1) and Q(1, 7) (b) $(x+5)^2 + (y-5)^2 = 40$

4.(a) Total area of the sheet $= 2xy \Rightarrow 2xy = 24 \Rightarrow y = \dfrac{12}{x}$

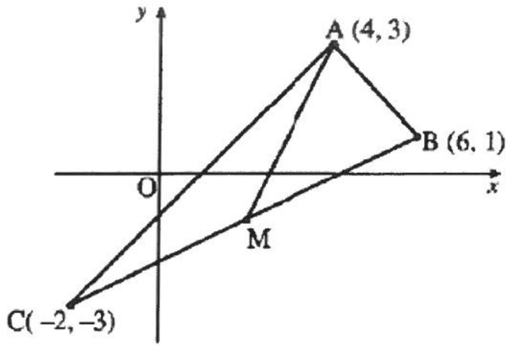
$L = 3x + 4y = \ldots = 3x + \dfrac{48}{x}$

(b) (i) $x = 4$ (ii) min cost = £198

Mixed Revision 5 (Non-Calculator)

1. (Grade C)

A triangle ABC has vertices A(4, 3), B(6, 1) and C(–2, –3) as shown in the diagram. Find the equation of AM, the median from A.



2. (Grade C)

Find the equation of the tangent at the point (3, 4) on the circle $x^2 + y^2 + 2x - 4y - 15 = 0$.

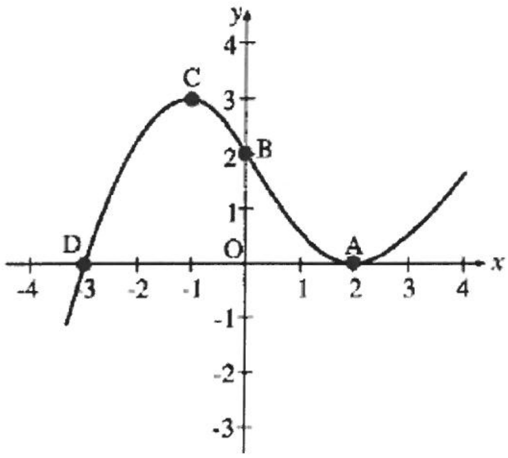
3. (Grade C)

Part of the graph of $y = f(x)$ is shown in the diagram.
On separate diagrams sketch the graphs of

(i) $y = f(x-1)$

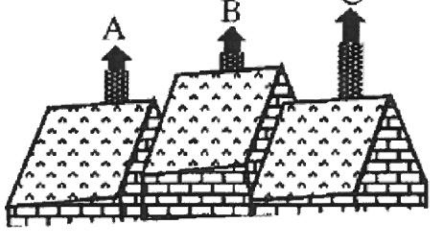
(ii) $y = -f(x) - 2$

indicating on each graph the images of A, B, C and D.



4. (Grade C)

Relative to a suitable set of axes, the tops of three chimneys have coordinates given by A(1, 3, 2), B(2, –1, 4) and C(4, –9, 8).
Show that A, B and C are collinear.



5. (Grade C)

A curve, for which $\frac{dy}{dx} = 6x^2 - 2x$, passes through the point (–1, 2).
Express y in terms of x.

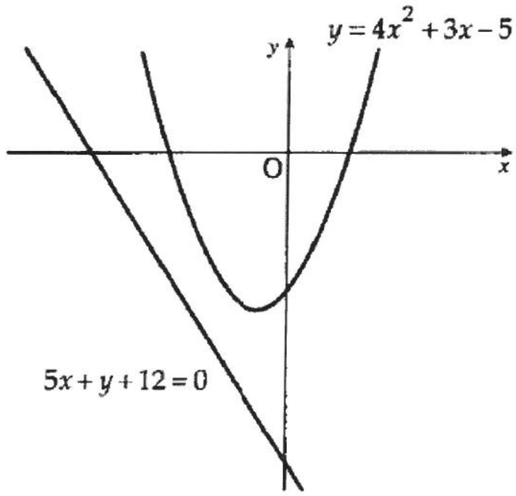
9. (Grade A/B)

(*a*) Show that the function $f(x) = 2x^2 + 8x - 3$ can be written in the form $f(x) = a(x+b)^2 + c$ where a, b and c are constants.

(*b*) Hence, or otherwise, find the coordinates of the turning point of the function f.

10. (Grade A/B)

The diagram below shows a parabola with equation $y = 4x^2 + 3x - 5$ and a straight line with equation $5x + y + 12 = 0$.
A tangent to the parabola is drawn parallel to the given straight line.
Find the x-coordinate of the point of contact of this tangent.



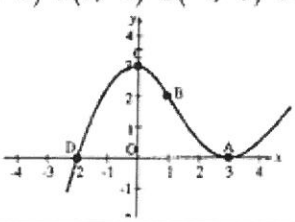
ANSWERS (Mixed Revision 5)

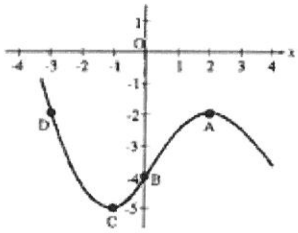
1. $y = 2x - 5$

2. $2x + y = 10$

3.

- •[1] translation of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- •[2] $A(3,0)$ $B(1,2)$ $C(0,3)$ $D(-2,0)$
- •[3] reflect in x − axis
- •[4] translation of $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$
- •[5] $A(2,-2)$ $B(0,-4)$ $C(-1,-5)$ $D(-3,-2)$





4. $\overrightarrow{BC} = 2\overrightarrow{AB}$, so $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ and hence A, B and C are collinear since B is a common point.

5. $y = 2x^3 - x^2 + 5$

6.(a) P(1, −2) and Q(3, 4) or vice-versa

(b) show that $m_{PT} \times m_{QT} = -1$ or $\overrightarrow{PT} . \overrightarrow{QT} = 0$

7. Area $= \dfrac{27}{4}$ units^2

HIGHER MATHEMATICS

Mixed Revision 6 (Calculator)

1. (Grade C)

A gardener feeds her trees weekly with "Bioforce, the wonder plant food". It is known that in a week the amount of plant food in the tree falls by about 25%.

The trees contain no Bioforce initially and the gardener applies 1g of Bioforce to each tree every Saturday.

(i) Write down a recurrence relation for the amount of plant food in the tree immediately after feeding.

(ii) If the level of Bioforce in the tree exceeds 5g, it will cause leaf burn. Is it safe to continue feeding the trees at this rate indefinitely?

2. (Grade C)

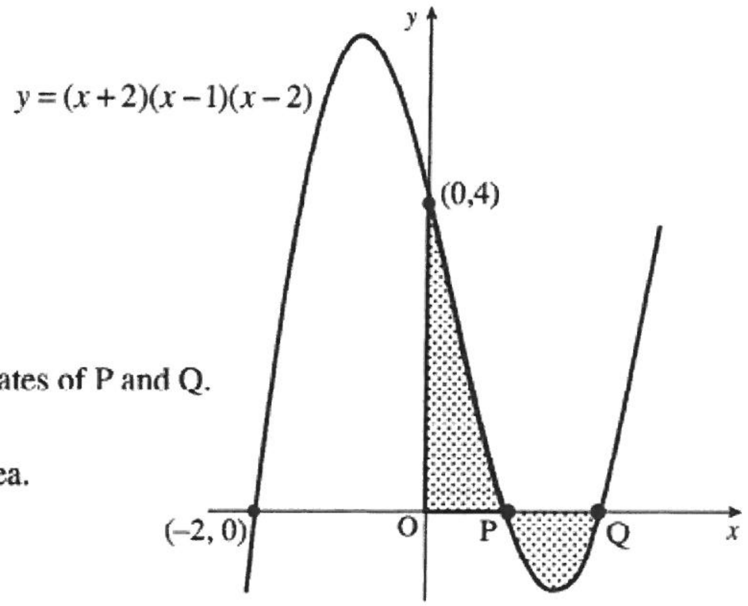
The graph of the curve with equation $y = 2x^3 + x^2 - 13x + a$ crosses the x-axis at the point (2,0).

(a) Find the value of a and hence write down the coordinates of the point at which this curve crosses the y-axis.

(b) Find algebraically the coordinates of the other points at which the curve crosses the x-axis.

5. (Grade C)

The diagram shows a sketch of the graph of $y = (x+2)(x-1)(x-2)$. The graph cuts the axes at (–2, 0), (0, 4) and the points P and Q.



(*a*) Write down the coordinates of P and Q.

(*b*) Find the total shaded area.

6. (Grade A/B)

Solve the equation $\cos 2x° + 2\sin x° = \sin^2 x°$ in the interval $0 \le x < 360$.

ANSWERS (Mixed Revision 6)

1.(i) $u_{n+1} = 0 \cdot 75u_n + 1 \quad (u_0 = 0)$

(ii) Since $-1 < 0 \cdot 75 < 1$, a limit exists and the limit is 4g.
It is therefore safe to continue feeding the trees at this rate indefinitely since the limit of 4g is less than the safe level of 5g.

2.(a) $a = 6$; (0, 6) (b) (−3, 0) and $\left(\frac{1}{2}, 0\right)$

3.(a) $2x + y = 10$ (b) D(4, 2) (c) 5 units^2

4.(a) $y = -2x - 1$ (b) $26 \cdot 6°$ [*hint*: use $m = \tan\theta$]

5.(a) P(1, 0) and Q(2, 0) (b) $2\frac{1}{2}$ units^2

6. $x = 90$, 199·5, 340.5 [*hint*: replace $\cos 2x$ with $1 - 2\sin^2 x$]

7.(a) $y = -5x - 3$; B(−1, 2) (b) $1\frac{1}{3}$ units^2

8.(a) proof [*hint*: $A = x^2 + 4xy$ and then use $x^2y = 500$ to replace y]

(b) 10 m by 10 m by 5 m

Mixed Revision 7 (Non-Calculator)

1. (Grade C)

Functions f and g, defined on suitable domains, are given by $f(x) = x^2 + 1$ and $g(x) = 1 - 2x$.

Find:

(*a*) $g(f(x))$;

(*b*) $g(g(x))$.

2. (Grade C)

If $y = \frac{1}{x^3} - \cos 2x,\ x \neq 0$, find $\frac{dy}{dx}$.

3. (Grade C)

Using the fact that $\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$, find the exact value of $\sin\left(\frac{7\pi}{12}\right)$.

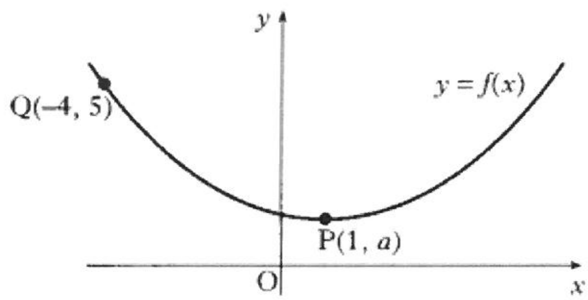
4. (Grade C)

The diagram shows the graph of a function $y = f(x)$.

Copy the diagram and on it sketch the graphs of:

(*a*) $y = f(x - 4)$;

(*b*) $y = 2 + f(x - 4)$.

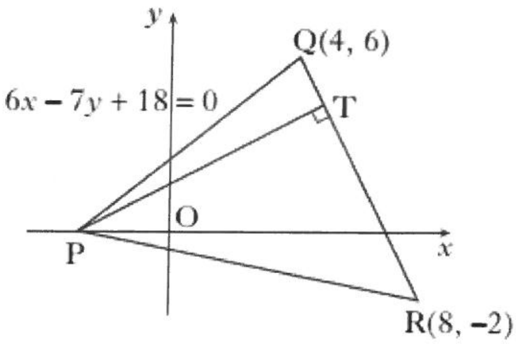


8. (Grade C)

Triangle PQR has vertex P on the x-axis, as shown in the diagram.

Q and R are the points (4, 6) and (8, –2) respectively.

The equation of PQ is $6x - 7y + 18 = 0$.



(*a*) State the coordinates of P.

(*b*) Find the equation of the altitude of the triangle from P.

(*c*) The altitude from P meets the line QR at T. Find the coordinates of T.

9. (Grade C)

Given that $y = \sqrt{3x^2 + 2}$, find $\dfrac{dy}{dx}$.

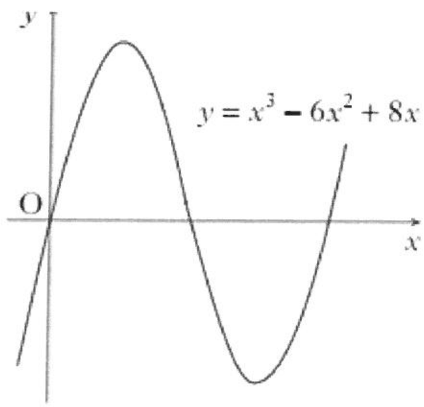
10. (Grade C)

Show that the line with equation $y = 6 - 2x$ is a tangent to the circle with equation $x^2 + y^2 + 6x - 4y - 7 = 0$ and find the coordinates of the point of contact of the tangent and the circle.

11. (Grade C)

The diagram shows a sketch of the curve with equation $y = x^3 - 6x^2 + 8x$.

Find the coordinates of the points on the curve where the gradient of the tangent is –1.

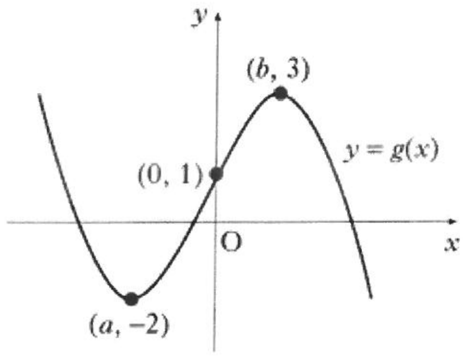


15. (Grade C)

The diagram shows the graph of $y = g(x)$.

(*a*) Sketch the graph of $y = -g(x)$.

(*b*) On the same diagram, sketch the graph of $y = 3 - g(x)$.



16. (Grade C)

Find the coordinates of the point on the curve $y = 2x^2 - 7x + 10$ where the tangent to the curve makes an angle of 45° with the positive direction of the x-axis.

17. (Grade C)

The curve $y = f(x)$ is such that $\frac{dy}{dx} = 4x - 6x^2$. The curve passes through the point $(-1, 9)$. Express y in terms of x.

18. (Grade C)

A curve has equation $y = 3x^2 - x^3$.

(*a*) Find the coordinates of the stationary points on this curve and determine their nature.

(*b*) State the coordinates of the points where the curve meets the coordinate axes and sketch the curve.

19. (Grade C)

$f(x) = x^3 - x^2 - 5x - 3$.

(*a*) (i) Show that $(x + 1)$ is a factor of $f(x)$.

(ii) Hence or otherwise factorise $f(x)$ fully.

(*b*) One of the turning points of the graph of $y = f(x)$ lies on the x-axis.

Write down the coordinates of this turning point.

20. (Grade C)

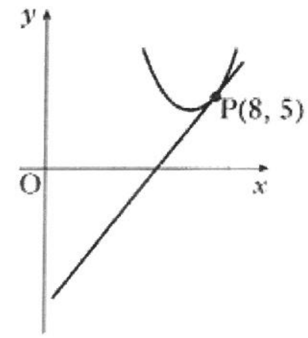
The circles with equations $(x - 3)^2 + (y - 4)^2 = 25$ and $x^2 + y^2 - kx - 8y - 2k = 0$ have the same centre.

Determine the radius of the larger circle.

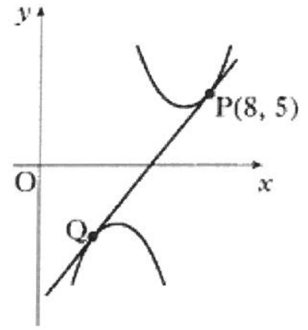
24. (Grade C)

The parabola with equation $y = x^2 - 14x + 53$ has a tangent at the point P(8, 5).

(*a*) Find the equation of this tangent.



(*b*) Show that the tangent found in (*a*) is also a tangent to the parabola with equation $y = -x^2 + 10x - 27$ and find the coordinates of the point of contact Q.



25. (Grade C)

Solve the equation $\sin 2x° = 6\cos x°$ for $0 \leq x \leq 360$.

26. (Grade C)

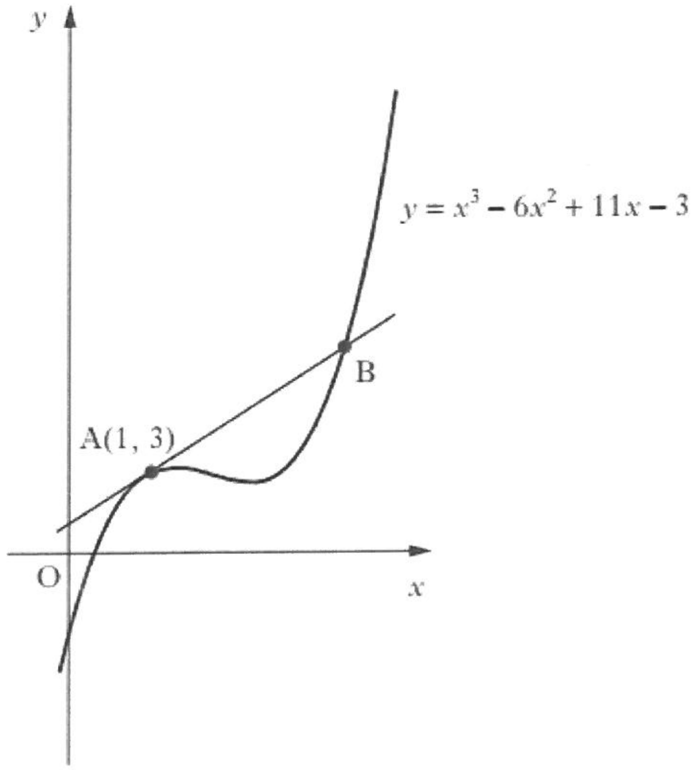
A function f is defined by $f(x) = (2x - 1)^5$.

Find the coordinates of the stationary point on the graph with equation $y = f(x)$ and determine its nature.

29. (Grade C)

(*a*) Show that $(x - 1)$ is a factor of $x^3 - 6x^2 + 9x - 4$ and hence factorise $x^3 - 6x^2 + 9x - 4$ fully.

(*b*) The diagram shows the graph with equation $y = x^3 - 6x^2 + 11x - 3$.



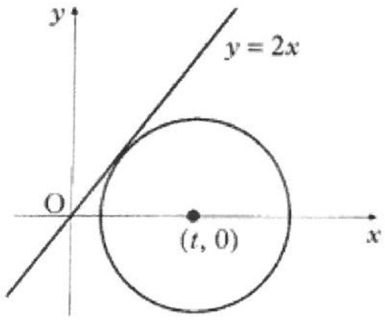
(i) Find the equation of the tangent to the curve $y = x^3 - 6x^2 + 11x - 3$ at the point A(1, 3).

(ii) Hence find the coordinates of B, the point of intersection of this tangent with the curve.

34. (Grade A/B)

(*a*) A circle has centre $(t, 0)$, $t > 0$, and radius 2 units.
Write down the equation of the circle.

(*b*) Find the exact value of t such that the line $y = 2x$ is a tangent to the circle.



ANSWERS (Mixed Revision 7)

1.(a) $g(f(x)) = -2x^2 - 1$ (b) $g(g(x)) = 4x - 1$

2. $\frac{dy}{dx} = -\frac{3}{x^4} + 2\sin 2x$ **3.** $\sin\left(\frac{7\pi}{12}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}}$

5. $\int \frac{4x^3 - 1}{x^2}\,dx = 2x^2 + \frac{1}{x} + C$

6.(a)(i) $f(g(x)) = x^2 + 8x + 19$ (ii) $g(f(x)) = x^2 + 7$

(b) The equation $f(g(x)) + g(f(x)) = 0$ simplifies to $2x^2 + 8x + 26 = 0$;
$b^2 - 4ac < 0$, hence the equation has no real roots

7.(b) $y = 2x - 3$ (c) $(1, -1)$ **8.** $y = 8x - 11$

9.(a) P(–3, 0) (b) $2y = x + 3$ (c) T(5, 4)

10. $\frac{dy}{dx} = \frac{3x}{\sqrt{3x^2 + 2}}$ **11.** point of contact = (1, 4)

12. (1, 3) and (3, –3)

13.(a) $g(f(x)) = 3x^3 - 2$ (c)(ii) $3x^3 + 4x^2 - 5x - 2 = (x-1)(3x+1)(x+2)$

(d) $x = -2$, $x = -\frac{1}{3}$, $x = 1$

30.(a) $y = 8x - 11$ (b) Q(4, −3)

31.(a) $\overrightarrow{AB} = 2\overrightarrow{BC}$, so $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ and hence A, B and C are collinear since B is a common point.

(b) D(5, 20, −9)

32. $x = 90, 270$

33. $\left(\frac{1}{2},0\right)$ is a (rising) point of inflexion

34.(b) $y = 3x - 3$ (c)(i) C(2, 3) (ii) $(x-2)^2 + (y-3)^2 = 10$

35.(a)(ii) $f(x) = (x-1)^2(2x+5)$ (b) $x = -\frac{5}{2}$, $x = 1$ (c) G(1, −1)

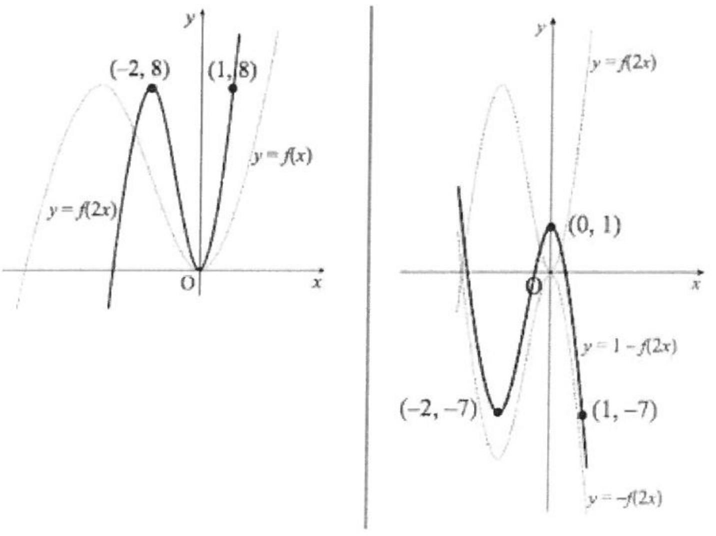
(d) $H\left(-\frac{5}{2},-8\right)$

36.(a) $x^3 - 6x^2 + 9x - 4 = (x-1)^2(x-4)$ (b) $y = 2x + 1$ (c) B(4, 9)

37.(a)(i) $\overrightarrow{DE} = 3\overrightarrow{EF}$, so $\overrightarrow{DE}$ is parallel to $\overrightarrow{EF}$ and hence D, E and F are collinear since E is a common point.

(ii) 3 : 1 (b) $k = 7$

38.



39.(a) $a = -1$, $b = -2$ (b) $f(x) = (x+1)(3x+2)(2x-1)$

40. $x = 0, 60, 300$

41. $x = 3$

42.(a) $f(g(x)) = 15 + 2x - x^2$ (b) $f(g(x)) = -(x-1)^2 + 16$

(c) $x = -3$, $x = 5$

43. $a = 2$

44.(a) $a = 4$, $b = 5$ (b) $x > 4$

45.