Higher
Mathematics

Sequences

Contents

CfE Edition

Sequences

1   Introduction to Sequences   A

> A **sequence** is an ordered list of objects (usually numbers).

Usually we are interested in sequences which follow a particular pattern. For example, $1, 2, 3, 4, 5, 6, \ldots$ is a sequence of numbers – the "…" just indicates that the list keeps going forever.

Writing a sequence in this way assumes that you can tell what pattern the numbers are following but this is not always clear, e.g.

$$28, \; 22, \; 19, \; 17\tfrac{1}{2}, \; \ldots.$$

For this reason, we prefer to have a formula or rule which explicitly defines the terms of the sequence.

It is common to use subscript numbers to label the terms, e.g.

$$u_1, \; u_2, \; u_3, \; u_4, \; \ldots$$

so that we can use u_n to represent the nth term.

We can then define sequences with a formula for the nth term. For example:

Formula	List of terms
$u_n = n$	1, 2, 3, 4, …
$u_n = 2n$	2, 4, 6, 8, …
$u_n = \tfrac{1}{2}n(n+1)$	1, 3, 6, 10, …
$u_n = \cos\left(\tfrac{n\pi}{2}\right)$	0, −1, 0, 1, …

Notice that if we have a formula for u_n, it is possible to work out *any* term in the sequence. For example, you could easily find u_{1000} for any of the sequences above without having to list all the previous terms.

Recurrence Relations

Another way to define a sequence is with a **recurrence relation**. This is a rule which defines each term of a sequence using previous terms.

For example:

$$u_{n+1} = u_n + 2, \ u_0 = 4$$

says "the first term (u_0) is 4, and each other term is 2 more than the previous one", giving the sequence $4, 6, 8, 10, 12, 14, \ldots$.

Notice that with a recurrence relation, we need to work out all earlier terms in the sequence before we can find a particular term. It would take a long time to find u_{1000}.

Another example is interest on a bank account. If we deposit £100 and get 4% interest per year, the balance at the end of each year will be 104% of what it was at the start of the year.

$$u_0 = 100$$
$$u_1 = 104\% \text{ of } 100 = 1{\cdot}04 \times 100 = 104$$
$$u_2 = 104\% \text{ of } 104 = 1{\cdot}04 \times 104 = 108{\cdot}16$$
$$\vdots$$

The complete sequence is given by the recurrence relation

$$u_{n+1} = 1{\cdot}04 u_n \text{ with } u_0 = 100,$$

where u_n is the amount in the bank account after n years.

EXAMPLE

The value of an endowment policy increases at the rate of 5% per annum. The initial value is £7000.

(a) Write down a recurrence relation for the policy's value after n years.

(b) Calculate the value of the policy after 4 years.

(a) Let u_n be the value of the policy after n years.

So $u_{n+1} = 1{\cdot}05 u_n$ with $u_0 = 7000$.

(b) $u_0 = 7000$
$u_1 = 1{\cdot}05 \times 7000 = 7350$
$u_2 = 1{\cdot}05 \times 7350 = 7717{\cdot}5$
$u_3 = 1{\cdot}05 \times 7717{\cdot}5 = 8103{\cdot}375$
$u_4 = 1{\cdot}05 \times 8103{\cdot}375 = 8508{\cdot}54375$

After 4 years, the policy is worth £8508·54.

2   Linear Recurrence Relations

A

In Higher, we will deal with recurrence relations of the form

$$u_{n+1} = au_n + b$$

where a and b are any real numbers and u_0 is specified. These are called **linear recurrence relations** of order one.

Note

To properly define a sequence using a recurrence relation, we must specify the initial value u_0.

EXAMPLES

1. A patient is injected with 156 ml of a drug. Every 8 hours, 22% of the drug passes out of his bloodstream. To compensate, a further 25 ml dose is given every 8 hours.

    (a) Find a recurrence relation for the amount of drug in his bloodstream.

    (b) Calculate the amount of drug remaining after 24 hours.

    (a) Let u_n be the amount of drug in his bloodstream after $8n$ hours.

    $$u_{n+1} = 0{\cdot}78u_n + 25 \text{ with } u_0 = 156$$

    (b) $u_0 = 156$

    $$u_1 = 0{\cdot}78 \times 156 + 25 = 146{\cdot}68$$

    $$u_2 = 0{\cdot}78 \times 146{\cdot}68 + 25 = 139{\cdot}4104$$

    $$u_3 = 0{\cdot}78 \times 139{\cdot}4104 + 25 = 133{\cdot}7401$$

    After 24 hours, he will have 133.74 ml of drug in his bloodstream.

2. A sequence is defined by the recurrence relation $u_{n+1} = 0{\cdot}6u_n + 4$ with $u_0 = 7$.

    Calculate the value of u_3 and the smallest value of n for which $u_n > 9{\cdot}7$.

    $$u_0 = 7$$

    $$u_1 = 0{\cdot}6 \times 7 + 4 = 8{\cdot}2$$

    $$u_2 = 0{\cdot}6 \times 8{\cdot}2 + 4 = 8{\cdot}92$$

    $$u_3 = 0{\cdot}6 \times 8{\cdot}92 + 4 = 9{\cdot}352$$

    The value of u_3 is 9.352
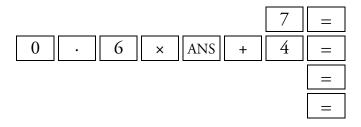
    $$u_4 = 9{\cdot}6112$$

    $$u_5 = 9{\cdot}76672$$

    The smallest value of n for which $u_n > 9{\cdot}7$ is 5

Using a Calculator

Using the ANS button on the calculator, we can carry out the above calculation more efficiently.

$$\boxed{0}\ \boxed{\cdot}\ \boxed{6}\ \boxed{\times}\ \boxed{\text{ANS}}\ \boxed{+}\ \boxed{\begin{matrix}7 \\ 4\end{matrix}}\ \boxed{\begin{matrix}= \\ = \\ = \\ =\end{matrix}}$$

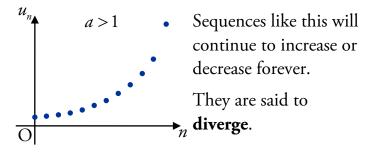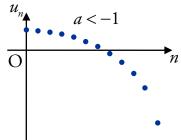## 3   Divergence and Convergence   A

If we plot the graphs of some of the sequences that we have been dealing with, then some similarities will occur.
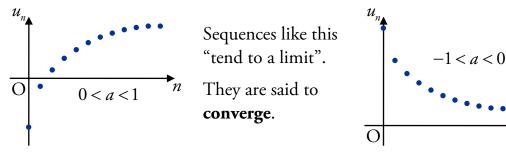
Divergence

Sequences defined by recurrence relations in the form $u_{n+1} = au_n + b$ where $a < -1$ or $a > 1$, will have a graph like this:

$a > 1$

Sequences like this will continue to increase or decrease forever.

They are said to **diverge**.

$a < -1$

Convergence

Sequences defined by recurrence relations in the form $u_{n+1} = au_n + b$ where $-1 < a < 1$, will have a graph like this:

Sequences like this "tend to a limit".

They are said to **converge**.

$0 < a < 1$

$-1 < a < 0$

4   The Limit of a Sequence   A

We saw that sequences defined by $u_{n+1} = au_n + b$ with $-1 < a < 1$ "tend to a limit". In fact, it is possible to work out this limit just from knowing a and b.

> The sequence defined by $u_{n+1} = au_n + b$ with $-1 < a < 1$ tends to a limit l as $n \to \infty$ (i.e. as n gets larger and larger) given by
> $$l = \frac{b}{1-a}.$$

You will need to know this formula, as it is not given in the exam.

EXAMPLES

1. The deer population in a forest is estimated to drop by 7·3% each year. Each year, 20 deer are introduced to the forest. The initial deer population is 200.

   (a) How many deer will there be in the forest after 3 years?

   (b) What is the long term effect on the population?

   (a) $u_{n+1} = 0 \cdot 927 u_n + 20$

   $u_0 = 200$

   $u_1 = 0 \cdot 927 \times 200 + 20 = 205 \cdot 4$

   $u_2 = 0 \cdot 927 \times 205 \cdot 4 + 20 = 210 \cdot 4058$

   $u_3 = 0 \cdot 927 \times 210 \cdot 4058 + 20 = 215 \cdot 0461$

   Therefore there are 215 deer living in the forest after 3 years.

   (b) A limit exists, since $-1 < 0 \cdot 927 < 1$.

   $l = \dfrac{b}{1-a}$ where $a = 0 \cdot 927$ and $b = 20$

   $ = \dfrac{20}{1 - 0 \cdot 927}$

   $ = 273 \cdot 97$ (to 2 d.p.).

   Therefore the number of deer in the forest will settle around 273.

   Note
   Whenever you calculate a limit using this method, you must state that "A limit exists since $-1 < a < 1$".

2. A sequence is defined by the recurrence relation $u_{n+1} = ku_n + 2k$ and the first term is u_0.

   Given that the limit of the sequence is 27, find the value of k.

The limit is given by $\dfrac{b}{1-a} = \dfrac{2k}{1-k}$, and so

$$\frac{2k}{1-k} = 27$$
$$27(1-k) = 2k$$
$$29k = 27$$
$$k = \frac{27}{29}.$$

5   Finding a Recurrence Relation for a Sequence   A

If we know that a sequence is defined by a linear recurrence relation of the form $u_{n+1} = au_n + b$, and we know three consecutive terms of the sequence, then we can find the values of a and b.

This can be done easily by forming two equations and solving them simultaneously.

EXAMPLE

A sequence is defined by $u_{n+1} = au_n + b$ with $u_1 = 4$, $u_2 = 3{\cdot}6$ and $u_3 = 2{\cdot}04$.

Find the values of a and b.

Form two equations using the given terms of the sequence:

$$u_2 = au_1 + b \qquad \text{and} \qquad u_3 = au_2 + b$$
$$3{\cdot}6 = 4a + b \quad ① \qquad\qquad 2{\cdot}04 = 3{\cdot}6a + b \quad ②.$$

Eliminate b:

$$① - ②: \quad 1{\cdot}56 = 0{\cdot}4a$$
$$a = \frac{1{\cdot}56}{0{\cdot}4}$$
$$= 3{\cdot}9.$$

Put $a = 3{\cdot}9$ into ①:

$$4 \times 3{\cdot}9 + b = 3{\cdot}6$$
$$b = 3{\cdot}6 - 15{\cdot}6$$
$$b = -12.$$

So $a = 3{\cdot}9$ and $b = -12$.