SEQUENCES AND SERIES 1

SIGMA NOTATION

1. Evaluate:

(a) $\sum_{k=1}^{5}(k+1)$ (b) $\sum_{k=3}^{6}k^2$ (c $\sum_{k=0}^{4}\frac{1}{2}k$

(d) $\sum_{k=9}^{12}(3k-1)$ (e) $\sum_{k=0}^{5}2^k$ (f) $\sum_{k=1}^{6}k^3$

(g) $\sum_{k=1}^{4}k(k+2)$ (h) $\sum_{k=8}^{12}k(k-1)(k-2)$ (i) $\sum_{k=1}^{4}\frac{1}{k}$

(j) $\sum_{k=0}^{5}(2k^2+1)$ (k) $\sum_{k=4}^{10}(k+1)(2k+1)$ (l) $\sum_{k=2}^{6}(2k-3)(3k-1)$

(m) $\sum_{k=4}^{9}\frac{2520}{k+1}$ (n) $\sum_{k=1}^{4}(-3)^k$ (o) $\sum_{k=-6}^{-3}(2k+7)$

(p) $\sum_{k=0}^{6}4^{k+2}$ (q) $\sum_{k=1}^{5}(2k^3-k^2-4k+1)$

2. Define $S_n=\sum_{k=1}^{n}(2k-1)$, where n is a positive integer.

Then $S_1=\sum_{k=1}^{1}(2k-1)=2\times1-1=1$,

and $S_2=\sum_{k=1}^{2}(2k-1)=(2\times1-1)+(2\times2-1)=1+3=4$.

Evaluate S_3, S_4 and S_5, and conjecture a formula for S_n in terms of n.

3. Define $S_n=\sum_{k=1}^{n}\frac{1}{k(k+1)}$, where n is a positive integer.

Evaluate S_1, S_2, S_3, S_4 and S_5, and conjecture a formula for S_n in terms of n.

4. Repeat question 3 given $S_n=\sum_{k=1}^{n}(3k^2-3k+1)$.

ANSWERS

1. (a) 20 (b) 86 (c 5 (d) 122
(e) 63 (f) 441 (g) 50 (h) 3870
(i) $2\frac{1}{12}$ (j) 116 (k) 896 (l) 335
(m) 2131 (n) 60 (o) −8 (p) 87 376
(q) 340

2. $S_3=9$, $S_4=16$, $S_5=25$; $S_n=n^2$

3. $S_1=\frac{1}{2}$, $S_2=\frac{2}{3}$, $S_3=\frac{3}{4}$, $S_4=\frac{4}{5}$, $S_5=\frac{5}{6}$; $S_n=\frac{n}{n+1}$

4. $S_1=1$, $S_2=8$, $S_3=27$, $S_4=64$, $S_5=125$; $S_n=n^3$

[Note: Shortly you will be able to **prove** the conjectures in questions 2, 3 and 4.]

SEQUENCES AND SERIES 2

USE OF PARTIAL FRACTIONS

1. Express $\frac{1}{(k+1)(k+2)}$ in partial fractions.

   Hence find $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)}$ and $\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)}$.

2. Express $\frac{1}{k(k+1)}$ in partial fractions.

   Hence find $\sum_{k=1}^{n} \frac{1}{k(k+1)}$ and $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$.

3. Express $\frac{2}{(2k-1)(2k+1)}$ in partial fractions.

   Hence find $\sum_{k=1}^{n} \frac{2}{(2k-1)(2k+1)}$ and $\sum_{k=1}^{\infty} \frac{2}{(2k-1)(2k+1)}$.

4. Express $\frac{1}{k(k-1)}$ in partial fractions.

   Hence find $\sum_{k=2}^{n} \frac{1}{k(k-1)}$ and $\sum_{k=2}^{\infty} \frac{1}{k(k-1)}$.

5. Express $\frac{1}{(2k+1)(2k+3)}$ in partial fractions.

   Hence find $\sum_{k=1}^{n} \frac{1}{(2k+1)(2k+3)}$ and $\sum_{k=1}^{\infty} \frac{1}{(2k+1)(2k+3)}$.

6. Express $\frac{3}{(3k-1)(3k+2)}$ in partial fractions.

   Hence find $\sum_{k=1}^{n} \frac{3}{(3k-1)(3k+2)}$ and $\sum_{k=1}^{\infty} \frac{3}{(3k-1)(3k+2)}$.

7. Express $\frac{4}{(4k-3)(4k+1)}$ in partial fractions.

   Hence find $\sum_{k=1}^{n} \frac{4}{(4k-3)(4k+1)}$ and $\sum_{k=1}^{\infty} \frac{4}{(4k-3)(4k+1)}$.

8. Use partial fractions to show that

   $$\sum_{k=1}^{n} \frac{1}{(k+1)(k+3)} = \frac{5}{12} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}.$$

   Hence find $\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+3)}$.

9. Use partial fractions to show that

   $$\sum_{k=1}^{n} \frac{2}{k(k+2)} = \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}.$$

   Hence find $\sum_{k=1}^{\infty} \frac{2}{k(k+2)}$.

10. Use partial fractions to show that

    $$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+3)} = \frac{1}{3} - \frac{1}{4(2n+1)} - \frac{1}{4(2n+3)}.$$

    Hence find $\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+3)}$.

11. Express $\frac{1}{k^2-1}$ in partial fractions.

    Hence: (a) show that $\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}$, and find $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$;

    (b) find the sum of the series
    $$\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \ldots + \frac{1}{399}.$$

ANSWERS

1. $\dfrac{1}{(k+1)(k+2)} = \dfrac{1}{k+1} - \dfrac{1}{k+2}$;

   $\displaystyle\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} = \frac{1}{2} - \frac{1}{n+2}$; $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)} = \frac{1}{2}$

2. $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$; $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$; $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$

3. $\dfrac{2}{(2k-1)(2k+1)} = \dfrac{1}{2k-1} - \dfrac{1}{2k+1}$;

   $\displaystyle\sum_{k=1}^{n} \frac{2}{(2k-1)(2k+1)} = 1 - \frac{1}{2n+1}$; $\displaystyle\sum_{k=1}^{\infty} \frac{2}{(2k-1)(2k+1)} = 1$

4. $\dfrac{1}{k(k-1)} = -\dfrac{1}{k} + \dfrac{1}{k-1}$; $\displaystyle\sum_{k=2}^{n} \frac{1}{k(k-1)} = 1 - \frac{1}{n}$; $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(k-1)} = 1$

5. $\dfrac{1}{(2k+1)(2k+3)} = \dfrac{1/2}{2k+1} - \dfrac{1/2}{2k+3}$;

   $\displaystyle\sum_{k=1}^{n} \frac{1}{(2k+1)(2k+3)} = \frac{1}{6} - \frac{1}{2(2n+3)}$; $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k+1)(2k+3)} = \frac{1}{6}$

6. $\dfrac{3}{(3k-1)(3k+2)} = \dfrac{1}{3k-1} - \dfrac{1}{3k+2}$;

   $\displaystyle\sum_{k=1}^{n} \frac{3}{(3k-1)(3k+2)} = \frac{1}{2} - \frac{1}{3n+2}$; $\displaystyle\sum_{k=1}^{\infty} \frac{3}{(3k-1)(3k+2)} = \frac{1}{2}$

7. $\dfrac{4}{(4k-3)(4k+1)} = \dfrac{1}{4k-3} - \dfrac{1}{4k+1}$;

   $\displaystyle\sum_{k=1}^{n} \frac{4}{(4k-3)(4k+1)} = 1 - \frac{1}{4n+1}$; $\displaystyle\sum_{k=1}^{\infty} \frac{4}{(4k-3)(4k+1)} = 1$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+3)} = \frac{5}{12}$

9. $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k(k+2)} = \frac{3}{2}$

10. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+3)} = \frac{1}{3}$

11. $\dfrac{1}{k^2-1} = \dfrac{1/2}{k-1} - \dfrac{1/2}{k+1}$;

    (a) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2-1} = \frac{3}{4}$

    (b) $\dfrac{1}{3} + \dfrac{1}{8} + \dfrac{1}{15} + \dfrac{1}{24} + \ldots + \dfrac{1}{399} = \dfrac{589}{840}$

SEQUENCES AND SERIES 3

THE METHOD OF DIFFERENCES

1. Show that $(k+1)^2 - k^2 = 2k+1$.

   Hence find: (a) $\sum_{k=1}^{n}(2k+1)$

   (b) the sum of the series $3+5+7+9+\ldots+45$.

2. Show that $\frac{1}{2}k(k+1) - \frac{1}{2}(k-1)k = k$.

   Hence find: (a) $\sum_{k=1}^{n} k$

   (b) the sum of the series $1+2+3+4+\ldots+80$.

3. Show that $\frac{1}{6}k(k+1)(2k+1) - \frac{1}{6}(k-1)k(2k-1) = k^2$.

   Hence find: (a) $\sum_{k=1}^{n} k^2$

   (b) the sum of the series $1+4+9+16+\ldots+2500$.

4. Show that $\frac{1}{4}k^2(k+1)^2 - \frac{1}{4}(k-1)^2k^2 = k^3$.

   Hence find: (a) $\sum_{k=1}^{n} k^3$

   (b) the sum of the series $1+8+27+64+\ldots+3375$.

5. Show that $\frac{1}{3}k(k+1)(k+2) - \frac{1}{3}(k-1)k(k+1) = k(k+1)$.

   Hence find: (a) $\sum_{k=1}^{n} k(k+1)$

   (b) the sum of the series $(1\times2)+(2\times3)+(3\times4)+(4\times5)+\ldots+(35\times36)$.

6. Show that $(k+1)^2(k+2) - k^2(k+1) = (k+1)(3k+2)$.

   Hence find: (a) $\sum_{k=1}^{n}(k+1)(3k+2)$

   (b) the sum of the series $(2\times5)+(3\times8)+(4\times11)+(5\times14)+\ldots+(45\times134)$.

7. Show that $\frac{1}{4}(k+1)(k+2)(k+3)(k+4) - \frac{1}{4}k(k+1)(k+2)(k+3) = (k+1)(k+2)(k+3)$.

   Hence find: (a) $\sum_{k=1}^{n}(k+1)(k+2)(k+3)$

   (b) the sum of the series $(2\times3\times4)+(3\times4\times5)+(4\times5\times6)+\ldots+(18\times19\times20)$.

8. Show that $(k+1)^3 = k^3+3k^2+3k+1$, and find a similar expansion for $(k-1)^3$.

   Hence show that $\frac{1}{2}(k+1)^3 - \frac{1}{2}(k-1)^3 = 3k^2+1$.

   Find: (a) $\sum_{k=1}^{n}(3k^2+1)$

   (b) the sum of the series $4+13+28+49+\ldots+676$.

* 9. Show that $\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)} = \frac{2}{k(k+1)(k+2)}$.

   Hence show that $\sum_{k=1}^{n}\frac{2}{k(k+1)(k+2)} = \frac{1}{2} - \frac{1}{(n+1)(n+2)}$ and find $\sum_{k=1}^{\infty}\frac{2}{k(k+1)(k+2)}$.

* 10. Show that $\frac{1}{k^2} - \frac{1}{(k+1)^2} = \frac{2k+1}{k^2(k+1)^2}$.

   Hence show that $\sum_{k=1}^{n}\frac{2k+1}{k^2(k+1)^2} = 1 - \frac{1}{(n+1)^2}$.

   Find: (a) the sum of the series $\frac{3}{1^2\times2^2} + \frac{5}{2^2\times3^2} + \frac{7}{3^2\times4^2} + \ldots + \frac{25}{12^2\times13^2}$

   (b) $\sum_{k=1}^{\infty}\frac{2k+1}{k^2(k+1)^2}$.

ANSWERS

1. (a) $n(n+2)$ (b) 528

2. (a) $\frac{1}{2}n(n+1)$ (b) 3240

3. (a) $\frac{1}{6}n(n+1)(2n+1)$ (b) 42 925

4. (a) $\frac{1}{4}n^2(n+1)^2$ (b) 14 400

5. (a) $\frac{1}{3}n(n+1)(n+2)$ (b) 15 540

6. (a) $(n+1)^2(n+2)-2$ (b) 93 148

7. (a) $\frac{1}{4}(n+1)(n+2)(n+3)(n+4)-6$ (b) 35 904

8. (a) $\frac{1}{2}n(2n^2+3n+3)$ (b) 3735

9. $\sum_{k=1}^{\infty} \frac{2}{k(k+1)(k+2)} = \frac{1}{2}$

10. (a) $\frac{168}{169}$ (b) 1

SEQUENCES AND SERIES 4

USE OF STANDARD FORMULAE (1)

Throughout this exercise, the following standard formulae may be quoted and used without proof.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}; \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}; \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

1. Using the above standard formulae, evaluate:

(a) $\sum_{k=1}^{15} k$ (b) $\sum_{k=1}^{20} k^2$ (c) $\sum_{k=1}^{9} k^3$

(d) $\sum_{k=1}^{12} (2k^2 + 3k - 1)$ (e) $\sum_{k=1}^{16} (4k^2 + 2k - 3)$ (f) $\sum_{k=1}^{8} k^2(4k+1)$

(g) $\sum_{k=1}^{10} (k^3 + 3k^2 - 5k + 2)$ (h) $\sum_{k=1}^{28} (2k^3 - k^2 + 2k + 5)$

2. Write each of the following series in the form $\sum_{k=1}^{n} f(k)$ and, using the standard formulae, evaluate each summation.

(a) $(1\times2)+(2\times3)+(3\times4)+(4\times5)+\ldots+(29\times30)$

(b) $(1\times3)+(2\times4)+(3\times5)+(4\times6)+\ldots+(20\times22)$

(c) $(1\times5)+(2\times6)+(3\times7)+(4\times8)+\ldots+(15\times19)$

(d) $(1\times3)+(2\times5)+(3\times7)+(4\times9)+\ldots+(30\times61)$

(e) $(1\times1)+(2\times4)+(3\times7)+(4\times10)+\ldots+(25\times73)$

(f) $(1\times7)+(2\times9)+(3\times11)+(4\times13)+\ldots+(45\times95)$

(g) $(1\times11)+(2\times15)+(3\times19)+(4\times23)+\ldots+(16\times71)$

(h) $(1^2\times3)+(2^2\times4)+(3^2\times5)+(4^2\times6)+\ldots+(20^2\times22)$

3. Using the standard formulae, find fully factorised expressions for each summation below.

(a) $\sum_{k=1}^{n} (2k+3)$ (b) $\sum_{k=1}^{n} (4k+1)$ (c) $\sum_{k=1}^{n} (2k^2-1)$

(d) $\sum_{k=1}^{n} (3k^2-5)$ (e) $\sum_{k=1}^{n} (k-1)(k+1)$ (f) $\sum_{k=1}^{n} (3k-2)$

(g) $\sum_{k=1}^{n} (2k^2-5)$ (h) $\sum_{k=1}^{n} (6k^2-1)$ (i) $\sum_{k=1}^{n} (2k^2+3)$

✱ 4. Study the worked example below carefully.

Worked Example

Evaluate $\sum_{k=25}^{45} k^2$.

Solution

$$\begin{aligned}\sum_{k=25}^{45} k^2 &= 25^2+26^2+27^2+\ldots+45^2\\ &= (1^2+2^2+3^2+\ldots+45^2)-(1^2+2^2+3^2+\ldots+24^2)\\ &= \sum_{k=1}^{45} k^2 - \sum_{k=1}^{24} k^2\\ &= \frac{45\times46\times91}{6}-\frac{24\times25\times49}{6}\\ &= 31395-4900\\ &= 26495\end{aligned}$$

Using a similar approach, evaluate:

(a) $\sum_{k=36}^{60} k$ (b) $\sum_{k=15}^{30} k^2$ (c) $\sum_{k=16}^{24} k^3$

ANSWERS

1. (a) 120 (b) 2870 (c) 2025 (d) 1522
   (e) 6208 (f) 5388 (g) 3925 (h) 322 910

2. (a) 8990 (b) 3290 (c) 1720 (d) 19 375
   (e) 15 925 (f) 67 965 (g) 6936 (h) 49 840

3. (a) $n(n+4)$ (b) $n(2n+3)$ (c) $\frac{n(n+2)(2n-1)}{3}$
   (d) $\frac{n(n+3)(2n-3)}{2}$ (e) $\frac{n(n-1)(2n+5)}{6}$ (f) $\frac{n(3n-1)}{2}$
   (g) $\frac{n(n-2)(2n+7)}{3}$ (h) $n^2(2n+3)$ (i) $\frac{n(2n^2+3n+10)}{3}$

4. (a) 1200 (b) 8440 (c) 75 600

SEQUENCES AND SERIES 5

USE OF STANDARD FORMULAE (2)

Throughout this exercise, the following standard formulae may be quoted and used without proof.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}; \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}; \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

1. Find an expression for $\sum_{k=1}^{n} k(k+1)$, giving your answer in fully factorised form.
   Hence sum the series $(1\times2)+(2\times3)+(3\times4)+(4\times5)+\ldots+(28\times29)$.

2. Find an expression for $\sum_{k=1}^{n} k(k+3)$, giving your answer in fully factorised form.
   Hence sum the series $(1\times4)+(2\times5)+(3\times6)+(4\times7)+\ldots+(35\times38)$.

3. Find an expression for $\sum_{k=1}^{n} k^2(k+1)$, giving your answer in fully factorised form.
   Hence sum the series $(1^2\times2)+(2^2\times3)+(3^2\times4)+(4^2\times5)+\ldots+(15^2\times16)$.

4. Find an expression for $\sum_{k=1}^{n} k(k+1)(k+2)$, giving your answer in fully factorised form.
   Hence sum the series
   $(1\times2\times3)+(2\times3\times4)+(3\times4\times5)+(4\times5\times6)+\ldots+(20\times21\times22)$.

5. Find an expression for $\sum_{k=1}^{n} k(k+1)(k+3)$, giving your answer in fully factorised form.
   Hence sum the series
   $(1\times2\times4)+(2\times3\times5)+(3\times4\times6)+(4\times5\times7)+\ldots+(42\times43\times45)$.

6. Find an expression for $\sum_{k=1}^{n} k(k+1)^2$, giving your answer in fully factorised form.
   Hence sum the series $(1\times2^2)+(2\times3^2)+(3\times4^2)+(4\times5^2)+\ldots+(20\times21^2)$.

7. Find an expression for $\sum_{k=1}^{n} 2k(2k-1)$, giving your answer in fully factorised form.
   Hence sum the series $(2\times1)+(4\times3)+(6\times5)+(8\times7)+\ldots+(50\times49)$.

8. Find an expression for $\sum_{k=1}^{n} k(k+1)(2k-1)$, giving your answer in fully factorised form.
   Hence sum the series
   $(1\times2\times1)+(2\times3\times3)+(3\times4\times5)+(4\times5\times7)+\ldots+(18\times19\times35)$.

9. Find an expression for $\sum_{k=1}^{n} k(k^2+1)$, factorising your answer as far as possible.
   Hence sum the series $(1\times2)+(2\times5)+(3\times10)+(4\times17)+\ldots+(12\times145)$.

10. Find an expression for $\sum_{k=1}^{n} k(3k-2)^2$, giving your answer in fully factorised form.
    Hence sum the series $(1\times1^2)+(2\times4^2)+(3\times7^2)+(4\times10^2)+\ldots+(24\times4900)$.

ANSWERS

1. $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$; 8120
2. $\sum_{k=1}^{n} k(k+3) = \frac{n(n+1)(n+5)}{3}$; 16 800
3. $\sum_{k=1}^{n} k^2(k+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$; 15 640
4. $\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$; 53 130
5. $\sum_{k=1}^{n} k(k+1)(k+3) = \frac{n(n+1)(n+2)(3n+13)}{12}$; 920 458
6. $\sum_{k=1}^{n} k(k+1)^2 = \frac{n(n+1)(n+2)(3n+5)}{12}$; 50 050
7. $\sum_{k=1}^{n} 2k(2k-1) = \frac{n(n+1)(4n-1)}{3}$; 21 450
8. $\sum_{k=1}^{n} k(k+1)(2k-1) = \frac{n(n+1)(n+2)(3n-1)}{6}$; 60 420
9. $\sum_{k=1}^{n} k(k^2+1) = \frac{n(n+1)(n^2+n+2)}{4}$; 6162
10. $\sum_{k=1}^{n} k(3k-2)^2 = \frac{n^2(n+1)(9n-7)}{4}$; 752 400

SEQUENCES AND SERIES 6

ARITHMETIC SEQUENCES

1. Find the required term in each of these arithmetic sequences.

(a) 10, 12, 14, 16, … (100^{th} term) (b) 1, 4, 7, 10, … (16^{th} term)
(c) 3, 5, 7, 9, … (20^{th} term) (d) 14, 10, 6, 2, … (15^{th} term)
(e) -5, -1, 3, 7, … (12^{th} term) (f) 10, 5, 0, -5, … (21^{st} term)

2. Find a formula for the nth term, u_n, of each of these arithmetic sequences.

(a) 1, 3, 5, 7, … (b) 4, 7, 10, 13, …
(c) 1, 5, 9, 13, … (d) 5, 7, 9, 11, …
(e) 2, 5, 8, 11, … (f) 13, 8, 3, -2, …

3. Find the first term and the common difference for each arithmetic sequence described below.

(a) The 5^{th} term is 21 and the 10^{th} term is 41.
(b) The 3^{rd} term is 12 and the 8^{th} term is −18.
(c) The 4^{th} term is 14 and the 15^{th} term is 47.
(d) The 4^{th} term is −9 and the 15^{th} term is −31.
(e) The 6^{th} term is 23 and the 20^{th} term is 65.
(f) The 4^{th} term is 20 and the 9^{th} term is 8.

4. The 5^{th} term of an arithmetic sequence is 22 and the 17^{th} term is 52. Find the 45^{th} term of this sequence.

5. For the arithmetic sequence 1, 5, 9, 13, …, find a formula for u_n, the nth term, and hence find:

(a) the 10^{th} term
(b) which term is 285
(c) which term is the first to exceed 150.

6. For the arithmetic sequence 7, 12, 17, 22, …, find a formula for u_n, the nth term, and hence find:

(a) the 30^{th} term
(b) which term is 227
(c) which term is the first to exceed 1000.

7. For the arithmetic sequence −24, -21.5, -19, -16.5, …, find a formula for u_n, the nth term, and hence find:

(a) the 50^{th} term
(b) which term is 156
(c) which term is the first to exceed 1000.

8. For the arithmetic sequence 843, 836, 829, 822, …, find a formula for u_n, the nth term, and hence find:

(a) the 25^{th} term
(b) which term is 360
(c) which term is the first to be negative.

9. For the arithmetic sequence 56.3, 55.4, 54.5, 53.6, …, find a formula for u_n, the nth term, and hence find:

(a) the 40^{th} term
(b) which term is −27.4
(c) which term is the first to be negative.

10. The first term of an arithmetic sequence is 3, and the 6^{th} term is twice the 3^{rd} term. Find the common difference and the 10^{th} term.

11. An accountant has a salary scheme as outlined below.

STARTING SALARY £16, 000.
ANNUAL INCREASES OF £1, 000 GUARANTEED THEREAFTER.

Let £u_n denote the accountant's salary at the beginning of the nth year, after any increases which are then due have been added.

(a) Write down the values of u_1, u_2, u_3 and u_4.
(b) Find a formula for u_n.
(c) Find the value of n for which $u_{n+1} = 1.04u_n$ (that is, corresponding to an annual increase of 4%).

ANSWERS

1. (a) 208 (b) 46 (c) 41 (d) -42 (e) 39 (f) -90

2. (a) $u_n = 2n - 1$ (b) $u_n = 3n + 1$ (c) $u_n = 4n - 3$
   (d) $u_n = 2n + 3$ (e) $u_n = 3n - 1$ (f) $u_n = -5n + 18$

3. (a) $a = 5$, $d = 4$ (b) $a = 24$, $d = -6$ (c) $a = 5$, $d = 3$
   (d) $a = -3$, $d = -2$ (e) $a = 8$, $d = 3$
   (f) $a = 27.2$, $d = -2.4$ $\left[a = 27\frac{1}{5}, d = -2\frac{2}{5}\right]$

4. 122

5. $u_n = 4n - 3$; (a) 37 (b) u_{72} (c) u_{39}

6. $u_n = 5n + 2$; (a) 152 (b) u_{45} (c) u_{200}

7. $u_n = 2.5n - 26.5$; (a) 98.5 (b) u_{73} (c) u_{411}

8. $u_n = -7n + 850$; (a) 675 (b) u_{70} (c) u_{122}

9. $u_n = -0.9n + 57.2$; (a) 21.2 (b) u_{94} (c) u_{64}

10. $d = 3$; $u_{10} = 30$

11. (a) $u_1 = 16000$, $u_2 = 17000$, $u_3 = 18000$, $u_4 = 19000$
    (b) $u_n = 1000n + 15000$
    (c) $n = 10$

SEQUENCES AND SERIES 7

ARITHMETIC SERIES

1. Find the sum of each of these arithmetic series.

(a) $1 + 3 + 5 + 7 + \ldots$ (to 20 terms)
(b) $2 + 3 + 4 + 5 + \ldots$ (to 40 terms)
(c) $80 + 70 + 60 + 50 + \ldots$ (to 12 terms)
(d) $3{\cdot}5 + 3{\cdot}7 + 3{\cdot}9 + 4{\cdot}1 + \ldots$ (to 16 terms)
(e) $2 + 6 + 10 + 14 + \ldots$ (to 30 terms)
(f) $-4 - 5 - 6 - 7 - \ldots$ (to 18 terms)
(g) $4{\cdot}5 + 6 + 7{\cdot}5 + 9 + \ldots$ (to 25 terms)
(h) $15 + 13 + 11 + 9 + \ldots$ (to 16 terms)

2. Find the sum of each of the following arithmetic series.

(a) $2 + 4 + 6 + 8 + \ldots + 146$ (b) $3 + 8 + 13 + 18 + \ldots + 98$
(c) $100 + 95 + 90 + 85 + \ldots + 15$ (d) $4 + 10 + 16 + 22 + \ldots + 322$

3. (a) Find the sum of all the even natural numbers between 1 and 100 inclusive, i.e. $2 + 4 + 6 + 8 + \ldots + 100$.

(b) Find the sum of all the natural numbers between 1 and 100 which are divisible by 3, i.e. $3 + 6 + 9 + 12 + \ldots + 99$.

(c) Find the sum of all the 3-digit natural numbers which are divisible by 5, i.e. $100 + 105 + 110 + 115 + \ldots + 995$.

4. For the arithmetic series $5 + 7 + 9 + 11 + \ldots$, find a formula for S_n, the sum of the first n terms.
Hence find how many terms must be taken to give a sum of 192.

5. For the arithmetic series $2 + 5 + 8 + 11 + \ldots$, find a formula for S_n, the sum of the first n terms.
Hence find how many terms must be taken to give a sum of 876.

6. For the arithmetic series $3 + 10 + 17 + 24 + \ldots$, find a formula for S_n, the sum of the first n terms.
Hence find how many terms must be taken to give a sum of 2353.

7. For the arithmetic series $3 + 7 + 11 + 15 + \ldots$, find a formula for S_n, the sum of the first n terms.
Hence find how many terms must be taken to give a sum of 6555.

8. Find a formula for S_n, the sum of the first n terms of the arithmetic series
$$3 + 8 + 13 + 18 + \ldots$$
Hence find the least number of terms that are required to make a sum exceeding 2000.

9. Find a formula for S_n, the sum of the first n terms of the arithmetic series
$$6 + 7.5 + 9 + 10.5 + \ldots$$
Hence find the least number of terms that are required to give a sum exceeding 1000.

10. A woman wins a prize in the form of a guaranteed annual income for the rest of her life.
The income will be £1000 in the first year and will increase by £500 per annum thereafter, so that the income in the second year will be £1500, and so on.

If £S_n denotes the total income received over the first n years, find a formula for S_n, and hence find:
(a) the total income received over the first 10 years
(b) the number of years until the total income first exceeds £100 000.

11. A man wins a prize in the form of a guaranteed annual income for the rest of his life.
The income will be £10 000 in the first year and will increase by £2000 per annum thereafter, so that the income in the second year will be £12 000, and so on.

If £S_n denotes the total income received over the first n years, find a formula for S_n, and hence find:
(a) the total income received over the first 20 years
(b) the number of years until the total income first exceeds £1 000 000.

Find also the total income received in the 10^{th} year to the 15^{th} year (both years inclusive).

12. For the arithmetic series $5 + 8 + 11 + 14 + \ldots$, find the sum of the series from the 20^{th} term to the 40^{th} term (both terms inclusive).

* 13. In an arithmetic sequence, the 8^{th} term is twice the 4^{th} term.

(a) If a is the first term and d is the common difference, show that $a = d$.
(b) Given also that the 20^{th} term is 40, find the sum of the first 50 terms in this sequence.

* 14. The sum of the first 10 terms of an arithmetic sequence is 120.

(a) If a is the first term and d is the common difference, show that $2a + 9d = 24$.
(b) Given also that the sum of the first 20 terms of this sequence is 840, form another equation in a and d, and hence find a and d.

* 15. An arithmetic sequence has a common difference d.

If the sum of the first 20 terms of this sequence is 25 times the first term, find an expression for the first term in terms of d.

Find also an expression for the sum of the first 30 terms in terms of d only.

* 16. The formula for S_n, the sum of the first n terms of a particular arithmetic sequence, is $S_n = n^2 + 4n$. Let u_n denote the nth term of this sequence.

(a) Use the formula for S_n to find the sum of the terms in this sequence from the 8th to the 20th (both terms inclusive), i.e. $u_8 + u_9 + u_{10} + \ldots + u_{20}$

(b) It is required to find a formula for u_n.

(i) Find S_1, S_2 and S_3, and hence write down the values of the first three terms u_1, u_2 and u_3.
State the first term and common difference of this sequence, and hence find a formula for u_n.

(ii) Make use of the fact that $u_n = S_n - S_{n-1}$ to find a formula for u_n.

* 17. Find a simple formula for S_n, the sum of the first n terms of the arithmetic sequence 2, 5, 8, 11, …

Hence find the value of n for which the sum of the first $2n$ terms will exceed the sum of the first n terms by 224.

[Hint: Form the equation $S_{2n} = S_n + 224$.]

ANSWERS

1. (a) 400 (b) 860 (c) 300 (d) 80 (e) 1800
   (f) -225 (g) 562.5 (h) 0

2. (a) 5402 (b) 1010 (c) 1035 (d) 8802

3. (a) 2550 (b) 1683 (c) 98 550

4. $S_n = n(n+4)$; 12 terms

5. $S_n = \frac{n(3n+1)}{2}$; 24 terms

6. $S_n = \frac{n(7n-1)}{2}$; 26 terms

7. $S_n = n(2n+1)$; 57 terms

8. $S_n = \frac{n(5n+1)}{2}$; 29 terms

9. $S_n = \frac{n(1.5n+10.5)}{2}\left[= \frac{n(3n+21)}{4}\right]$; 34 terms

10. $S_n = 250n(n+3)$; (a) £32 500 (b) 19 years

11. $S_n = 1000n(n+9)$; (a) £580 000 (b) 28 years; £198 000

12. 1932

13. (b) 2550

14. (b) $2a + 19d = 84$; $a = -15$, $d = 6$

15. $a = 38d$; $S_{30} = 1575d$

16. (a) 403
    (b) (i) $S_1 = 5$, $S_2 = 12$, $S_3 = 21$; $u_1 = 5$, $u_2 = 7$, $u_3 = 9$;
    $a = 5$, $d = 2$; $u_n = 2n + 3$
    (ii) $u_n = 2n + 3$

17. $S_n = \frac{n(3n+1)}{2}$; $n = 7$

SEQUENCES AND SERIES 8

GEOMETRIC SEQUENCES

1. Find the required term in each of these geometric sequences.

(a) 1, 3, 9, 27, ... (10^{th} term)	(b) 4, 20, 100, 500, ... (8^{th} term)
(c) 2400, 1200, 600, 300, ... (7^{th} term)	(d) 1, -2, 4, -8, ... (15^{th} term)
(e) $2\frac{1}{4}, 1\frac{1}{2}, 1, \frac{2}{3}, \ldots$ (7^{th} term)	(f) $45, 15, 5, 1\frac{2}{3}, \ldots$ (6^{th} term)
(g) 1 000 000, 1 500 000, 2 250 000, 3 375 000, ... (7^{th} term)	
(h) $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \ldots$ (9^{th} term)	(i) 4, -12, 36, -108, ... (12^{th} term)

2. A geometric sequence of positive terms has first term 6 and third term 24. Find the eighth term of this sequence.

3. A geometric sequence has second term 6 and fifth term 162. Find the tenth term of this sequence.

4. The third term of a geometric sequence is 2, and the sixth term is 16. Find the twentieth term of this sequence.

5. A geometric sequence of alternating positive and negative terms has first term 8 and fifth term $\frac{1}{2}$.
   Find the common ratio and the tenth term of this sequence.

6. A geometric sequence has third term 6 and eighth term 1458. Find the twelfth term of this sequence.

7. A geometric sequence has second term 24 and seventh term 24 576. Find the fourth term of this sequence.

8. A geometric sequence of positive terms has fourth term 1000 and sixth term 160. Find the seventh term of this sequence.

9. The legendary wise man, as a reward for solving some problem, was offered whatever he chose to name. His request, which the King thought very modest, was "One grain of rice on the first square of a chessboard, two on the second, four on the third, eight on the fourth, and so on, with the number of grains doubling each time."

   Calculate the number of grains of rice needed for the last square, given that the chessboard is 8 squares by 8 squares.

Please turn over for question 10.

10. A piece of newspaper 0.005 cm thick is torn in two and the pieces placed on top of one another. These pieces are then torn in two and the four pieces placed on top of one another. This process of tearing is then continued.

    (a) If u_n denotes the thickness of the pile, in centimetres, <u>after</u> n tears, write down the values of u_1, u_2, u_3 and u_4, and find a formula for u_n.

    (b) Calculate the thickness after 8 tears.

ANSWERS

1. (a) 19, 683 (b) 312, 500 (c) 37.5 (d) 16, 384
   (e) $\frac{16}{81}$ (f) 11, 390, 625 (g) $\frac{5}{27}$ (h) $\frac{1}{768}$
   (i) $-708, 588$

2. 768 3. 39, 366 4. 262, 144

5. $r = -\frac{1}{2}$; $u_{10} = -\frac{1}{64}$

6. 118, 098 7. 384 8. 64

9. 2^{63} $(= 9.22 \times 10^{18})$

10. (a) $u_1 = 0.01$, $u_2 = 0.02$, $u_3 = 0.04$, $u_4 = 0.08$; $u_n = 0.01 \times 2^{n-1}$
    (b) 1.28 cm

SEQUENCES AND SERIES 9

GEOMETRIC SERIES

1. Find the sum of each of these geometric series.

   (a) $1 + 2 + 4 + 8 + \ldots$ (to 15 terms)
   (b) $2 + 6 + 18 + 54 + \ldots$ (to 10 terms)
   (c) $2 - 4 + 8 - 16 + \ldots$ (to 12 terms)
   (d) $64 + 32 + 16 + 8 + \ldots$ (to 9 terms)

2. Noting that $\sum_{k=1}^{20}(1.05)^k = 1.05 + (1.05)^2 + (1.05)^3 + (1.05)^4 + \ldots$ (to 20 terms), evaluate $\sum_{k=1}^{20}(1.05)^k$, giving your answer correct to 2 decimal places.

3. Evaluate:

   (a) $\sum_{k=1}^{50}(1.01)^k$, giving your answer correct to 2 decimal places
   (b) $\sum_{k=1}^{100}(1.1)^k$, giving your answer correct to the nearest whole number
   (c) $\sum_{k=0}^{30}(0.99)^k$, giving your answer correct to 2 decimal places.

* 4. Evaluate:

   (a) $\sum_{k=1}^{25}(1.06)^{2k}$, giving your answer correct to 2 decimal places
   (b) $\sum_{k=1}^{10}(0.8)^{3k-1}$, giving your answer correct to 3 decimal places.

5. Find the sum of each of these geometric series.

   (a) $3 + 6 + 12 + 24 + \ldots + 6144$
   (b) $5 + 15 + 45 + 135 + \ldots + 98\,415$

6. A geometric sequence of positive terms has first term 9 and third term 36. Find the sum of the first 10 terms of this sequence.

7. A geometric sequence has third term 54 and sixth term 1458. Find the sum of the first 8 terms of this sequence.

8. A geometric sequence has sixth term $\frac{32}{33}$ and seventh term $1\frac{31}{33}$.

   Find the sum of the first 10 terms of this sequence.

9. A geometric sequence has fifth term 18 and eighth term $\frac{2}{3}$.

   Find the sum of the first 6 terms of this sequence.

10. A geometric sequence has second term 2000 and seventh term $62\frac{1}{2}$.

    Find the sum of the first 8 terms of this sequence.

11. A child negotiates a new pocket-money deal with her father in which she will receive 1 pence on the first day of a month, 2 pence on the second day, 4 pence on the third day, 8 pence on the fourth day, and so on, with the amount doubling each day until the end of the month.

    How much would the child receive in total during the course of a month of 30 days? (Give your answer to the nearest million pounds!)

12. For the geometric sequence 3, 6, 12, 24, …:

    (a) find an expression for the nth term
    (b) find an expression for the sum of the first n terms
    (c) find how many terms must be taken to give a sum of 786 429.

13. Find an expression for the sum of the first n terms of the geometric sequence 1, 2, 4, 8, …

    Hence find:
    (a) the number of terms which must be taken to give a sum of 4095
    (b) the least number of terms which must be taken to give a sum exceeding 100 000.

14. Find an expression for the sum of the first n terms of the geometric sequence 10, 30, 90, 270, …

    Hence find:
    (a) the number of terms which must be taken to give a sum of 32 800
    (b) the least number of terms which must be taken to give a sum exceeding 1 000 000.

15. A geometric sequence has first term 5 and fourth term 40.

    Find an expression for the sum of the first n terms of this sequence, and hence find the least number of terms which must be taken to give a sum exceeding 100 000.

Questions 16 and 17 are on the next page.

16. A line is divided into six parts whose length form a geometric sequence. Let u_1 denote the length of the shortest part, u_2 the length of the second shortest part, and so on, with u_6 denoting the length of the longest part.

Given that the shortest part has length 3 cm and the longest part has length 96 cm, find the length of the whole line.

17. A line 5115 units long is divided into n parts such that the lengths of the parts form a geometric sequence.

Given that the shortest two parts have lengths 5 units and 10 units, find the value of n.

ANSWERS

1. (a) 32, 767 (b) 59, 048 (c) -2730 (d) $127\frac{3}{4}$

2. 34.72

3. (a) 65.11 (b) 151, 576 (c) 26.77

4. (a) 158.36 (b) 1.310

5. (a) 12, 285 (b) 147, 620

6. 9207 7. 19, 680 8. 31

9. 2184 10. 7968.75

11. £11 million (i.e. £11, 000, 000)

12. (a) $u_n = 3 \times 2^{n-1}$ (b) $S_n = 3(2^n - 1)$ (c) 18

13. $S_n = 2^n - 1$; (a) 12 (b) 17

14. $S_n = 5(3^n - 1)$; (a) 8 (b) 12

15. $S_n = 5(2^n - 1)$; 15 terms

16. 189 cm 17. $n = 10$

SEQUENCES AND SERIES 10

INFINITE GEOMETRIC SERIES

1. Explain why a sum to infinity exists for each of the geometric series below, and find the sum to infinity in each case.

(a) $16 + 8 + 4 + 2 + \ldots$ (b) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$

(c) $4 + 1 + \frac{1}{4} + \frac{1}{16} + \ldots$ (d) $84 - 42 + 21 - 10\frac{1}{2} + \ldots$

(e) $9.6 + 7.2 + 5.4 + 4.05 + \ldots$ (f) $0.1 + 0.05 + 0.025 + 0.0125 + \ldots$

(g) $2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \ldots$ (h) $125 + 25 + 5 + 1 + \ldots$

(i) $80 + 13\frac{1}{3} + 2\frac{2}{9} + \frac{10}{27} + \ldots$ (j) $10 - 9 + 8.1 - 7.29 + \ldots$

2. The first term of a geometric series is 2, and the sum to infinity of the series is 4. Find the common ratio and the sixth term of this series.

3. A geometric series has a common ratio of $\frac{2}{5}$, and the sum to infinity of the series is 25.
Find the first and fourth terms of this series.

4. The common ratio of a geometric series is $-\frac{2}{3}$, and the sum to infinity of the series is -12.
Find the first three terms of this series.

5. Find the sum to infinity of the geometric series having a second term of -9 and a fifth term of $\frac{1}{3}$.

6. u_1, u_2, u_3, ... is a geometric sequence of positive terms. The first term, u_1, is equal to 60, and the fifth term, u_5, is equal to $3\frac{3}{4}$.

Evaluate $\sum_{k=1}^{\infty} u_k$.

7. A geometric series has a common ratio of $-\frac{1}{4}$ and a sum to infinity of 8.
Find the third term of this series.

8. The first term of a geometric series is 100, and the third term is 1.
Find the sum to infinity of each of the two possible series.

9. The nth term, u_n, of a geometric series is given by the formula $u_n = \frac{1}{4^n}$.

(a) Find the first five terms of this series.
(b) State the common ratio of this series and explain why the series has a sum to infinity.
(c) Find the sum to infinity.

10. Repeat question 9 for the geometric series whose nth term, u_n, is given by the formula $u_n = \frac{1}{2^{2n-1}}$.

✳ 11. Study the worked example below carefully.

Worked Example

For the infinite geometric series $1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \ldots$, find the range of values of x for which the series will have a sum to infinity, and find an expression for the sum to infinity when it exists.

Solution

$a = 1$, $\quad r = \frac{x/2}{1} = \frac{x}{2}$

S_∞ will exist when $-1 < r < 1 \Rightarrow -1 < \frac{x}{2} < 1$
$\Rightarrow -2 < x < 2$.

When $-2 < x < 2$, $S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{x}{2}} = \frac{2}{2-x}$.

Repeat the working of the above example for each of the infinite geometric series below.

(a) $x + x^2 + x^3 + x^4 + \ldots$ (b) $1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \ldots$

(c) $1 + 2x + 4x^2 + 8x^3 + \ldots$ (d) $1 + \frac{3x}{2} + \frac{9x^2}{4} + \frac{27x^3}{8} + \ldots$

(e) $\left(x+\frac{1}{2}\right) + \left(x+\frac{1}{2}\right)^2 + \left(x+\frac{1}{2}\right)^3 + \left(x+\frac{1}{2}\right)^4 + \ldots$

(f) $1 + (2x-1) + (2x-1)^2 + (2x-1)^3 + \ldots$

* It is given that

$$1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \frac{1}{1-x}, \qquad -1 < x < 1.$$

(a) Differentiate both sides of this identity to show that

$$1 + 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n + \ldots = \frac{1}{(1-x)^2}, \qquad -1 < x < 1.$$

(b) Make use of the series above to find an expression for the sum of the infinite series:

(i) $2 + 3x + 4x^2 + 5x^3 + \ldots + (n+2)x^n + \ldots, \qquad -1 < x < 1$

(ii) $x + 2x^2 + 3x^3 + 4x^4 + \ldots + nx^n + \ldots, \qquad -1 < x < 1.$

Give each expression in the form of a single algebraic fraction.

ANSWERS

1. (a) 32 (b) $1\frac{1}{2}$ (c) $5\frac{1}{3}$ (d) 56 (e) 38.4 (f) 0.2
   (g) 6 (h) 156.25 (i) 96 (j) $5\frac{5}{19}$

2. $r = \frac{1}{2}$; $u_6 = \frac{1}{16}$

3. $u_1 = 15$; $u_4 = \frac{24}{25}$

4. $u_1 = -20$; $u_2 = 13\frac{1}{3}$; $u_3 = -8\frac{8}{9}$

5. $20\frac{1}{4}$ 6. 120 7. $\frac{5}{8}$

8. $S_\infty = 111\frac{1}{9}$ or $S_\infty = 90\frac{10}{11}$

9. (a) $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \frac{1}{1024}$ (b) $r = \frac{1}{4}$; $-1 < r < 1$ (c) $\frac{1}{3}$

10. (a) $\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}, \frac{1}{512}$ (b) $r = \frac{1}{4}$; $-1 < r < 1$ (d) $\frac{2}{3}$

11. (a) When $-1 < x < 1$, $S_\infty = \frac{x}{1-x}$ (b) When $-3 < x < 3$, $S_\infty = \frac{3}{3-x}$
    (c) When $-\frac{1}{2} < x < \frac{1}{2}$, $S_\infty = \frac{1}{1-2x}$ (d) When $-\frac{2}{3} < x < \frac{2}{3}$, $S_\infty = \frac{2}{2-3x}$
    (e) When $-1\frac{1}{2} < x < \frac{1}{2}$, $S_\infty = \frac{2x+1}{1-2x}$ (f) When $0 < x < 1$, $S_\infty = -\frac{1}{2x}$

12. (b) (i) $\frac{2-x}{(1-x)^2}$ (ii) $\frac{x}{(1-x)^2}$

SEQUENCES AND SERIES 11

MISCELLANEOUS PROBLEMS

1. Express $\dfrac{1}{x^2+5x+6}$ in partial fractions.

   Hence find an expression for $\sum_{k=1}^{n}\dfrac{1}{k^2+5k+6}$.

   Evaluate $\sum_{k=1}^{\infty}\dfrac{1}{k^2+5k+6}$.

2. Write down formulae for $\sum_{k=1}^{n}k$, $\sum_{k=1}^{n}k^2$ and $\sum_{k=1}^{n}k^3$.

   Make use of these formulae to find an expression in terms of n for:

   (a) $\sum_{k=1}^{n}k(k-1)(k+1)$ (b) $\sum_{k=1}^{n}(k^3+3k^2+k)$

   (c) $\sum_{k=1}^{n}(2k^3+k^2-k)$ (d) $\sum_{k=1}^{n}k(k-1)(3k+1)$.

   Give each expression as a single fraction in fully factorised form.

3. For the arithmetic sequence 8, 14, 20, 26, …:
   (a) find a formula for u_n, the nth term, and hence find which term is 140
   (b) find a formula for S_n, the sum of the first n terms, and hence find how many terms must be taken to give a sum of 848.

4. An arithmetic sequence u_1, u_2, u_3, … is such that the first term, u_1, is 5, and the fourth term, u_4 is 29.
   (a) Find the tenth term, u_{10}.
   (b) Find an expression for $\sum_{k=1}^{n}u_k$, and hence find the least number of terms which must be taken to give a sum exceeding 10 000.

5. For the geometric sequence 5, 10, 20, 40, …:
   (a) find the 15th term
   (b) find the first term to exceed 1 000 000
   (c) find the sum of the first 12 terms.

6. Find a formula for S_n, the sum of the first n terms of the series

   $$7 + 9 + 11 + 13 + \ldots$$

   How many terms of the series must be taken to give a sum of 1755?

7. A geometric sequence of positive terms has first term $\frac{1}{2}$ and seventh term 32.

   Find: (a) the 18th term of this sequence
   (b) the sum of the first ten terms of this sequence.

8. Show that $5x$, $10x^3$ and $20x^5$, where $x \neq 0$, could be the first three terms of a geometric series, and state the common ratio of this series in terms of x.

   When $x = \frac{1}{3}$, show that the sum to infinity of the series exists, and find it.

9. Express $\dfrac{1}{(3x-1)(3x+2)}$ in partial fractions.

   Hence find an expression for $\sum_{k=1}^{n}\dfrac{1}{(3k-1)(3k+2)}$ in terms of n.

   Evaluate: (a) $\dfrac{1}{2\times5}+\dfrac{1}{5\times8}+\dfrac{1}{8\times11}+\dfrac{1}{11\times14}+\ldots\dfrac{1}{77\times80}$

   (b) $\sum_{k=1}^{\infty}\dfrac{1}{(3k-1)(3k+2)}$

10. A man is to receive annual income from a trust fund for the rest of his life. The income is to paid as described below.
    The income in the first year will be £2000;
    The income in the second year will be £2500;
    The income in the third year will be £3000;
    And so on, with the amount of income increasing by £500 each year.

    (a) Find an expression for the total amount of income the man will receive in n years.
    (b) How many more years will the man have to live in order to receive a total of £200 000 from this income?

11. A sequence u_1, u_2, u_3, … is defined by the recurrence relation $u_{n+1} = 3u_n$, with first term, u_1, equal to a.
    (a) What name is given to this type of sequence?
    (b) Given that the sum of the first eight terms of this sequence is 21 320, find the value of a.

12. A sequence v_1, v_2, v_3, … is defined by the recurrence relation $v_{n+1} = v_n + d$, with first term, v_1, equal to a.
    (a) What name is given to this type of sequence?
    (b) Given that the fourth term, v_4, is equal to 8, write down an equation in a and d.
    (c) Given also that $\sum_{k=1}^{15}v_k = 210$, write down a second equation in a and d.
    (d) Find the values of a and d.
    (e) Find the value of the eighth term, v_8.
    (f) Evaluate $\sum_{k=1}^{25}v_k$.

13. Let u_n denote the nth term of the sequence 2, 9, 16, 23, …
Evaluate the sum $u_{30} + u_{31} + u_{32} + u_{33} + \ldots + u_{40}$.

14. Explain why the infinite geometric series 500 + 300 + 180 + 108 + … has a sum to infinity, and find the value of the sum to infinity.

15. By recognising a connection with an arithmetic or a geometric series, evaluate each of the following sums.
(a) 5 + 12 + 19 + 26 + … + 180
(b) 10 + 20 + 40 + 80 + … + 640
(c) 1000 + 995 + 990 + 985 + … + 400

16. $u_1, u_2, u_3, \ldots, u_{11}$ is an arithmetic sequence of eleven terms. The first term, u_1, is equal to 1, and the last term, u_{11}, is equal to 61.
Find the sum of the eleven terms in the sequence.

17. In an arithmetic sequence with first term a and common difference d, the fifth term is three times the second term.
(a) Show that $2a - d = 0$.
(b) Given also that the sum of the first ten terms is 50, find the values of a and d, and find the sum of the first 20 terms.

18. The fourth term of an arithmetic sequence is 20 and the ninth term is 8.
Find the sum of the first 100 terms of this sequence.

19. The first term of an arithmetic sequence is 3.
Given that the sixth term is twice the third term, find the tenth term of this sequence.

20. The nth term, u_n, of an arithmetic sequence is given by the formula

$$u_n = \frac{1}{2}(3 - n).$$

(a) Find the values of the first five terms of this sequence, and state the value of the common difference between successive terms.
(b) Find the sum of the first 20 terms.

21. How many terms of the arithmetic series 1 + 3 + 5 + 7 + … are required to make a sum of 1521?

22. Find the least number of terms of the geometric series 8 + 12 + 18 + 27 + … that are required to make a sum exceeding 1 000 000.

23. $u_1, u_2, u_3, \ldots$ is a geometric sequence of positive terms. The third term, u_3, is equal to 4, and the fifth term, u_5, is equal to 8.
Show that the sum of the first ten terms of this sequence is given by $62(\sqrt{2} + 1)$.

24. Find a formula for S_n, the sum of the first n terms of the series
5 + 10 + 20 + 40 + …
Hence evaluate the sum $u_{10} + u_{11} + u_{12} + \ldots + u_{20}$, where u_n denotes the nth term of the series 5 + 10 + 20 + 40 + …

25. The second term of a geometric series is 8 and the fifth term is $3\frac{3}{8}$.
Show that this series has a sum to infinity, and find the exact value of the sum to infinity.

26. A child has just negotiated a new pocket-money deal with her mother.
On the first day of a month, the child will receive 1 pence;
On the second day, the child will receive 5 pence;
On the third day, the child will receive 9 pence;
And so on, with the amount increasing by 4 pence each day.

(a) Find an expression for the total amount, in pence, the child will receive after n days.
(b) After how many days will the total amount received first exceed £10?

27. $u_1, u_2, u_3, \ldots$ is a geometric sequence. The third term, u_3, is equal to 2, and the sixth term, u_6, is equal to 16.
Evaluate $\sum_{k=1}^{10} u_k$.

28. A sequence $u_1, u_2, u_3, \ldots$ is such that the nth term, u_n, is defined by the formula
$u_n = \ln(2^n)$.
(a) By considering $u_{n+1} - u_n$, or otherwise, show that the sequence is in fact an arithmetic sequence.
(b) Find the least number of terms of this sequence which must be taken to give a sum exceeding 1000.

29. The second term of an infinite geometric series is 10, and the series has a sum to infinity of 45.
Form a quadratic equation in r, the common ratio, and solve it to find the two possible values of the common ratio.

30. A circular disc of radius R is cut into twelve sectors such that the areas of the sectors are in arithmetic sequence. Let u_1 be the smallest area, u_2 the second smallest area, and so on, with u_{12} being the largest area.
Given that the area of the largest sector is twice that of the smallest sector, find the angle (in degrees) between the straight edges of the smallest sector.

31. Show that the terms 21, 27 and 29 are neither in arithmetic sequence nor in geometric sequence.

(a) A number x is added to each of the terms 21, 27 and 29. The resulting terms are then squared to give the terms $(21+x)^2$, $(27+x)^2$ and $(29+x)^2$.
If these terms are in arithmetic sequence, find the value of x.

(b) A number y is added to each of the terms 21, 27 and 29 to give the terms $21+y$, $27+y$ and $29+y$.
If these terms are in geometric sequence, find the value of y.

32. u_1, u_2, u_3, ... is an arithmetic sequence with a common difference of 0.5 between successive terms. The terms u_4, u_8 and u_{14} form the first three terms of a geometric sequence.
Find the value of the first term, u_1, of the arithmetic sequence, and find the value of the common ratio of the geometric sequence.

33. u_1, u_2, u_3, ... is an arithmetic sequence with first term, u_1, equal to a, and with non-zero common difference d between successive terms. The terms u_4, u_7 and u_{16} form the first three terms of a geometric sequence.

(a) Show that a and d satisfy the equation $2a+3d=0$.

(b) Given also that $\sum_{k=1}^{6} u_k = 12$, find the values of a and d, and find the common ratio of the geometric sequence.

34. The nth term, u_n, of a sequence is given by the formula $u_n = n^2 + n$.

(a) Write down the values of u_1, u_2, u_3 and u_4.

(b) Which term has value 132?

(c) Make use of the formulae for $\sum_{k=1}^{n} k$ and $\sum_{k=1}^{n} k^2$ to find an expression for $\sum_{k=1}^{n} u_k$ in terms of n, giving your answer as a single fraction in fully factorised form.

Hence: (i) find how many terms must be taken to give a sum of 4600

(ii) evaluate $\sum_{k=30}^{40} u_k$.

ANSWERS

: $\frac{1}{x^2+5x+6}=\frac{1}{x+2}-\frac{1}{x+3}$; $\sum_{k=1}^{n}\frac{1}{k^2+5k+6}=\frac{1}{3}-\frac{1}{n+3}$,
$\sum_{k=1}^{\infty}\frac{1}{k^2+5k+6}=\frac{1}{3}$

2. $\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$, $\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$, $\sum_{k=1}^{n}k^3=\frac{n^2(n+1)^2}{4}$;
(a) $\frac{n(n+1)(n+2)(n-1)}{4}$ (b) $\frac{n(n+1)^2(n+4)}{4}$
(c $\frac{n(n+1)(n+2)(3n-1)}{6}$ (d) $\frac{n(n+1)(n-1)(9n+10)}{12}$

3. (a) $u_n=6n+2$;   the 23[rd] term   (b) $S_n=n(3n+5)$;   16 terms

4. (a) $u_{10}=77$ (b) $\sum_{k=1}^{n}u_k=n(4n+1)$;   50 terms

5. (a) 81 920   (b) the 19[th] term   (c 20 475

6. $S_n=n(n+6)$;   39 terms

7. (a) 65 536   (b) $511\frac{1}{2}$

8. $r=2x^2$;
When $x=\frac{1}{3}$, $r=\frac{2}{9}$ and so $-1<r<1$; $S_\infty=2\frac{1}{7}$

9. $\frac{1}{(3x-1)(3x+2)}=\frac{1}{3(3x-1)}-\frac{1}{3(3x+2)}$; $\sum_{k=1}^{n}\frac{1}{(3k-1)(3k+2)}=\frac{1}{6}-\frac{1}{3(3n+2)}$;
(a) $\frac{13}{80}$ (b) $\frac{1}{6}$

10. (a) $250n(n+7)$ (b) 25 years

11. (a) geometric   (b) $a=6.5$

12. (a) arithmetic   (b) $a+3d=8$ (c $a+7d=14$ (or equivalent)
(d) $a=3.5$, $d=1.5$ (e) $v_8=14$ (f) 537.5

13. 2640

14. $r=\frac{3}{5}$ and so $-1<r<1$; $S_\infty=1250$

15. (a) 2405   (b) 1270   (c 84 700

16. 341

17. (b) $a=\frac{1}{2}$, $d=1$;   200

18. -9160 19. 30

20. (a) $1, \frac{1}{2}, 0, -\frac{1}{2}, -1$; $d=-\frac{1}{2}$ (b) -75

21. 39 terms   22. 28 terms

24. $S_n=5(2^n-1)$;   5 240 320

25. $r=\frac{3}{4}$ and so $-1<r<1$; $S_\infty=42\frac{2}{3}$

26. (a) $n(2n-1)$ (b) 23 days

27. $\sum_{k=1}^{10}u_k=511\frac{1}{2}$

28. (a) $u_{n+1}-u_n=\ln 2$, so there is a common difference of $\ln 2$ between successive terms
(b) 54 terms

29. $r=\frac{1}{3}$ or $r=\frac{2}{3}$ 30. 20°

31. $27-21\neq 29-27$, so the terms are not in arithmetic sequence;
$\frac{27}{21}\neq\frac{29}{27}$, so the terms are not in geometric sequence;
(a) $x=-22$ (b) $y=-30$

32. $u_1=2.5$; $r=1.5$

33. (b) $a=-3$, $d=2$; $r=3$

35. (a) $u_1=2$, $u_2=6$, $u_3=12$, $u_4=20$ (b) the 11[th] term
(c $\sum_{k=1}^{n}u_k=\frac{n(n+1)(n+2)}{3}$;   (i) 23 terms   (ii) $\sum_{k=30}^{40}u_k=13\,970$

ADVANCED HIGHER MATHEMATICS

SEQUENCES AND SERIES HOMEWORK 1

1. Express $\dfrac{1}{(k+2)(k+3)}$ in partial fractions.

   Hence find an expression for $\displaystyle\sum_{k=1}^{n}\frac{1}{(k+2)(k+3)}$ and evaluate $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(k+2)(k+3)}$.

2. Express $\dfrac{1}{(4k-1)(4k+3)}$ in partial fractions and find an expression for $\displaystyle\sum_{k=1}^{n}\frac{1}{(4k-1)(4k+3)}$.

   Hence: (a) evaluate the sum
   $$\frac{1}{3\times7}+\frac{1}{7\times11}+\frac{1}{11\times15}+\frac{1}{15\times19}+\ldots+\frac{1}{75\times79};$$
   (b) evaluate the sum of the infinite series
   $$\frac{1}{3\times7}+\frac{1}{7\times11}+\frac{1}{11\times15}+\frac{1}{15\times19}+\ldots$$

3. Express $\dfrac{1}{k^2+4k+3}$ in partial fractions.

   Hence show that $\displaystyle\sum_{k=1}^{n}\frac{1}{k^2+4k+3}=\frac{5}{12}-\frac{1}{2(n+2)}-\frac{1}{2(n+3)}$ and evaluate $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2+4k+3}$.

4. Write down an expression for $\displaystyle\sum_{k=1}^{n}k^2$ and hence find an expression for $\displaystyle\sum_{k=1}^{n}(2k^2-7)$, giving your answer as a single fraction in fully factorised form.

5. Write down expressions for $\displaystyle\sum_{k=1}^{n}k$ and $\displaystyle\sum_{k=1}^{n}k^2$.

   Hence find an expression for $\displaystyle\sum_{k=1}^{n}k(k+2)$, giving your answer as a single fraction in fully factorised form, and evaluate
   $$(1\times3)+(2\times4)+(3\times5)+(4\times6)+\ldots+(38\times40).$$

Please turn over for questions 6, 7 and 8.

6. Make use of the formulae for $\displaystyle\sum_{k=1}^{n}k$, $\displaystyle\sum_{k=1}^{n}k^2$ and $\displaystyle\sum_{k=1}^{n}k^3$ to find an expression for $\displaystyle\sum_{k=1}^{n}k(k+1)(k+5)$, giving your answer as a single fraction in fully factorised form.

   Hence evaluate $(1\times2\times6)+(2\times3\times7)+(3\times4\times8)+(4\times5\times9)+\ldots+(15\times16\times20)$.

7. Make use of the formulae for $\displaystyle\sum_{k=1}^{n}k$, $\displaystyle\sum_{k=1}^{n}k^2$ and $\displaystyle\sum_{k=1}^{n}k^3$ to find an expression for $\displaystyle\sum_{k=1}^{n}k(k+1)(2k-1)$, giving your answer as a single fraction in fully factorised form.

   Hence evaluate $(1\times2\times1)+(2\times3\times3)+(3\times4\times5)+(4\times5\times7)+\ldots+(23\times24\times45)$.

8. Given that $\displaystyle\sum_{k=1}^{n}k^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$, find an expression for $\displaystyle\sum_{k=1}^{n}k^2(3k^2+1)$, giving your answer as a single fraction in fully factorised form.

SEQUENCES AND SERIES HOMEWORK 2

1. For the arithmetic sequence 3, 7, 11, 15, ..., find a formula for u_n, the nth term.
   Hence find:
   (a) the 20th term;
   (b) which term is 235;
   (c) which term is the first term to exceed 500.

2. For the arithmetic sequence 100, 98.4, 96.8, 95.2, ..., find which term is the first to be negative, and state the value of this term.

3. The third term of an arithmetic sequence is 13 and the seventh term is 25.
   Find the 40th term of this sequence.

4. Let u_n denote the nth term of the sequence –5, -3, -1, 1, ...
   Evaluate:
   (a) $u_1 + u_2 + u_3 + u_4 + ... + u_{30}$
   (b) $u_{50} + u_{51} + u_{52} + u_{53} + ... + u_{100}$.

5. Evaluate the sum of the series 2 + 5 + 8 + 11 + ... + 440.

6. For the arithmetic sequence 8, 14, 20, 26, ..., find a simple expression for S_n, the sum of the first n terms.
   Hence find:
   (a) the sum of the first 40 terms
   (b) how many terms must be taken to give a sum of 13 000
   (c) the least number of terms which must be taken to give a sum exceeding 20 000.

7. A rod is divided into 20 pieces such that the lengths of the pieces form an arithmetic sequence. Let u_1 denote the length of the shortest piece, u_2 the length of the second shortest piece, and so on, with u_{20} being the length of the longest piece.

   Given that the shortest piece is 4 units long and that the longest piece is 11 times the length of the second shortest piece, find the length of the whole rod.

8. The first term of an arithmetic sequence is 18, and the fourth term is 9.
   Show that the sum of the first four terms of this sequence is equal to the sum of the first nine terms.

9. An arithmetic sequence with first term a and common difference d is such that the seventh term is twice the second term.
   Show that a and d satisfy the equation $a - 4d = 0$.

   Given also that the 15th term of this sequence is 54, write down another equation in a and d.

   Hence find:
   (a) the 11th term of this sequence
   (b) the sum of the first 50 terms of this sequence.

10. The second term of an arithmetic sequence is 7 times the fifth term, and the sum of the first six terms is 72.
    (a) Find the first term and common difference.
    (b) Find also the sum of the first 45 terms.

11. A man wins a prize in the form of a guaranteed annual income for the rest of his life. The income will be £5000 in the first year and will increase by £1500 per annum thereafter, so that the income in the second year will be £6500, and so on.

    If £S_n denotes the total income received over the first n years, find a formula for S_n, and hence find:
    (a) the total income received over the first 10 years
    (b) the number of years until the total income first exceeds 500 000.

    Find also the total income received in the 5th year to the 20th year (both years inclusive).

12. In an arithmetic sequence of n terms, the first term is 7 and the last term is 43.
    (a) If d is the common difference between successive terms, show that n and d satisfy the equation $(n-1)d = 36$.
    (b) Given that the sum of the terms in the sequence is 250, find the values of n and d.

SEQUENCES AND SERIES HOMEWORK 3

1. A geometric sequence begins 2, 6, 18, 54, …

   (a) Find the 9th term of this sequence.
   (b) Find the sum of the first 12 terms.

2. A geometric sequence begins 6, 12, 24, 48, …

   (a) Which term of this sequence first exceeds 1 000 000?
   (b) Find the sum of the terms in this sequence between the 10th and the 15th (with both these terms included).

3. Find the sum of the geometric series 5 + 15 + 54 + 135 + … + 885 735.

4. The fourth term of a geometric sequence is 192 and the sixth term is 3072. Given that this is a sequence of positive terms, find the tenth term of this sequence.

5. Let S_n denote the sum of the first n terms of the geometric series

$$9 + 18 + 36 + 72 + \dots$$

   (a) Find a formula for S_n.
   (b) How many terms must be taken to give a sum of 2295?

6. The third term of a geometric sequence is 75 and the sixth term is 9375. Find the sum of the first eight terms of this sequence.

7. Explain why the geometric series $\frac{4}{5}+\frac{8}{15}+\frac{16}{45}+\frac{32}{135}+\dots$ has a sum to infinity, and find the value of the sum to infinity.

8. The first term of a geometric series is 28 and the series has a sum to infinity of 16. Find the second and third terms of this sequence.

9. A rod 20 metres (i.e. 20 000 mm) in length is divided into ten pieces such that the lengths of the pieces form a geometric sequence. Let u_1 denote the length of the shortest piece, u_2 the length of the second-shortest piece, and so on, with u_{10} denoting the length of the longest piece.

   Given that the length of the longest piece is 512 times the length of the shortest piece, find the length of the shortest piece to the nearest millimetre.

10. A woman has just won a prize in the form of a guaranteed annual income for the rest of her life. The prize is to be paid as follows.

    The income in the first year will be 25 pence;
    The income in the second year will be 50 pence;
    The income in the third year will be £1;
    And so on, with the income doubling annually.

    Let £S_n denote the total income the woman will receive in n years.

    (a) Find a formula for S_n.
    (b) How many more years will the woman have to live to receive at least £1 000 000 in total from this income?

11. A sequence u_1, u_2, u_3, … is defined by the recurrence relation $u_{n+1} = -0.6u_n$, with first term, u_1, equal to a.

    (a) Given that $\sum_{k=1}^{\infty} u_k = 15$, find the value of a.
    (b) Find the fifth term, u_5, correct to two decimal places.
    (c Evaluate $\sum_{k=1}^{8} u_k$, giving the answer correct to two decimal places.

12. A line is divided into five parts such that the lengths of the parts form a geometric sequence. Let u_1 denote the shortest length, u_2 the second-shortest length, and so on, with u_5 being the longest length.

    Given that the shortest length is 4 cm and the longest length is 324 cm, find the length of the whole line.