MATRICES 1

ADDITION, SUBTRACTION, SCALAR MULTIPLICATION AND TRANSPOSES OF MATRICES

1. $A=\begin{pmatrix}2 & 1\\ 3 & -4\end{pmatrix}$, $B=\begin{pmatrix}3 & 2 & -4\\ -1 & 0 & 5\end{pmatrix}$, $C=\begin{pmatrix}1 & 0 & 3\\ 2 & 7 & 6\\ 5 & -1 & 4\\ 8 & 12 & -2\end{pmatrix}$ and $D=\begin{pmatrix}1\\ -3\\ 4\end{pmatrix}$.

(a) State the order of each matrix.

(b) Write down the entry in the first row and second column of matrix A.

(c) Write down the entry in the second row and first column of matrix B.

(d) Write down the entry in the third row and second column of matrix C.

(e) State the value of the entry:

(i) a_{21} (ii) b_{22} (iii) c_{23} (iv) d_{11} (v) b_{13}
(vi) c_{12}

2. State the order of each of the following matrices.

(a) $\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}$ (b) $\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$ (c) $\begin{pmatrix}0 & 1 & 0\\ 1 & 1 & -1\end{pmatrix}$

(d) $\begin{pmatrix}2 & 1 & 2 & 1\\ 2 & 3 & 5 & 2\\ 8 & 6 & -1 & 3\end{pmatrix}$ (e) $\begin{pmatrix}1 & -3 & 0\end{pmatrix}$ (f) $\begin{pmatrix}2 & 1 & 2\\ 0 & 0 & 1\\ 4 & 5 & 8\end{pmatrix}$

3. Find the values of x and y in each matrix equation.

(a) $\begin{pmatrix}3x\\ 12\end{pmatrix}=\begin{pmatrix}6\\ y\end{pmatrix}$ (b) $\begin{pmatrix}1 & 3\\ 4 & 2\end{pmatrix}=\begin{pmatrix}x & 1-y\\ 4 & 2x\end{pmatrix}$

(c) $\begin{pmatrix}x+y\\ 2x\end{pmatrix}=\begin{pmatrix}7\\ 8\end{pmatrix}$ (d) $\begin{pmatrix}2x+3y\\ x+2y\end{pmatrix}=\begin{pmatrix}8\\ 5\end{pmatrix}$

(e) $\begin{pmatrix}2 & 3x+y\\ 4x-3y & 5\end{pmatrix}=\begin{pmatrix}2 & 3\\ 17 & 5\end{pmatrix}$

4. Simplify:

(a) $\begin{pmatrix}2\\ 1\end{pmatrix}+\begin{pmatrix}5\\ 3\end{pmatrix}$ (b) $\begin{pmatrix}6\\ -8\end{pmatrix}+\begin{pmatrix}-1\\ 9\end{pmatrix}$

(c) $\begin{pmatrix}p+q\\ p-q\end{pmatrix}+\begin{pmatrix}-p\\ q\end{pmatrix}$ (d) $\begin{pmatrix}5k\\ -k\end{pmatrix}+\begin{pmatrix}-3k\\ 8k\end{pmatrix}+\begin{pmatrix}4\\ 1\end{pmatrix}$

(e) $\begin{pmatrix}4 & 1\\ 2 & 3\\ 4 & 5\end{pmatrix}+\begin{pmatrix}1 & 6\\ 2 & 5\\ 3 & 3\end{pmatrix}$ (f) $\begin{pmatrix}2x\\ y\end{pmatrix}+\begin{pmatrix}3x\\ -3y\end{pmatrix}$

(g) $\begin{pmatrix}3 & 0\\ 0 & 5\\ 4 & 1\end{pmatrix}+\begin{pmatrix}2 & 3\\ 1 & 4\\ 5 & 5\end{pmatrix}+\begin{pmatrix}-4 & 2\\ 3 & -3\\ 4 & -9\end{pmatrix}$ (h) $\begin{pmatrix}3 & 4 & 2\\ 2 & 1 & 6\end{pmatrix}+\begin{pmatrix}2 & 1 & -1\\ -3 & 0 & 4\end{pmatrix}$

(i) $\begin{pmatrix}4 & 1 & 3\\ 2 & 0 & 0\\ -2 & 1 & -1\end{pmatrix}+\begin{pmatrix}2 & 0 & 2\\ 1 & 0 & 0\\ 0 & 2 & -3\end{pmatrix}$ (j) $\begin{pmatrix}5 & -3 & 2\\ -2 & -1 & 3\\ 0 & 2 & -2\end{pmatrix}+\begin{pmatrix}-5 & 3 & -2\\ 2 & 1 & -3\\ 0 & -2 & 2\end{pmatrix}$

(k) $\begin{pmatrix}3\\ 2\end{pmatrix}-\begin{pmatrix}1\\ 3\end{pmatrix}$ (l) $\begin{pmatrix}4\\ -2\end{pmatrix}-\begin{pmatrix}-2\\ 6\end{pmatrix}$

(m) $\begin{pmatrix}4a\\ 2b\end{pmatrix}-\begin{pmatrix}a\\ -b\end{pmatrix}$ (n) $\begin{pmatrix}3m\\ -2n\end{pmatrix}+\begin{pmatrix}-m\\ 4n\end{pmatrix}-\begin{pmatrix}5m\\ 3n\end{pmatrix}$

(o) $\begin{pmatrix}2 & 5\\ 1 & -4\\ -1 & 3\end{pmatrix}-\begin{pmatrix}-2 & 4\\ 3 & 1\\ 4 & -1\end{pmatrix}$ (p) $\begin{pmatrix}-2 & 3\\ -1 & -1\end{pmatrix}+\begin{pmatrix}1 & -1\\ 2 & -2\end{pmatrix}$

(q) $\begin{pmatrix}-1 & 1\\ 1 & -1\\ -1 & 1\end{pmatrix}+\begin{pmatrix}3 & 1\\ -1 & 1\\ 3 & 1\end{pmatrix}-\begin{pmatrix}-3 & 1\\ 3 & -3\\ 3 & -1\end{pmatrix}$ (r) $\begin{pmatrix}-1\\ 2\\ 0\end{pmatrix}+\begin{pmatrix}-2\\ 3\\ 2\end{pmatrix}$

(s) $\begin{pmatrix}-2 & -4\\ 2 & -1\end{pmatrix}+\begin{pmatrix}0 & -1\\ 2 & 3\end{pmatrix}$ (t) $\begin{pmatrix}a+2b\\ 2a-b\end{pmatrix}-\begin{pmatrix}a+b\\ 3a-b\end{pmatrix}$

(u) $\begin{pmatrix}1\\ -2\\ 0\end{pmatrix}+\begin{pmatrix}2\\ 2\\ 1\end{pmatrix}$ (v) $\begin{pmatrix}-1 & 9 & -4\\ 2 & 3 & -2\\ 0 & -1 & 2\end{pmatrix}+\begin{pmatrix}-3 & 2 & 1\\ 0 & -4 & 3\\ -1 & 1 & -2\end{pmatrix}$

(w) $\begin{pmatrix}-2 & 3\\ -1 & -1\end{pmatrix}+\begin{pmatrix}3 & -3\\ 1 & 2\end{pmatrix}$ (x) $\begin{pmatrix}-1\\ 2\\ 0\end{pmatrix}-\begin{pmatrix}-2\\ 3\\ 2\end{pmatrix}$

(y) $\begin{pmatrix}-1 & 4 & -2\\ 2 & -1 & -1\\ 0 & 3 & 2\end{pmatrix}-\begin{pmatrix}-3 & 0 & 1\\ 0 & -2 & 3\\ -1 & 3 & -2\end{pmatrix}$ (z) $\begin{pmatrix}-3 & -4\\ -3 & -1\end{pmatrix}-\begin{pmatrix}0 & -1\\ 2 & 1\end{pmatrix}$

5. Simplify:

(a) $5\begin{pmatrix}1 & -4 & -6\\ 3 & -3 & -1\end{pmatrix}$ (b) $-3\begin{pmatrix}-1 & 3 & 1\\ -4 & 2 & 0\\ -5 & -1 & -3\end{pmatrix}$ (c) $2\begin{pmatrix}-3 & 0 & 1\\ 2 & 1 & -2\\ -3 & 1 & 1\end{pmatrix}$

(d) $3\begin{pmatrix}4 & 1 & -2\\ -1 & 0 & -2\\ 5 & 1 & 1\end{pmatrix}-2\begin{pmatrix}3 & -4 & -1\\ 3 & 2 & 0\\ 1 & 0 & -3\end{pmatrix}$

6. Given the matrices

$$A=\begin{pmatrix}1 & -1 & 0\\ 2 & 1 & 2\end{pmatrix},\ B=\begin{pmatrix}0 & 1 & -1\\ 3 & 2 & 1\end{pmatrix} \text{ and } C=\begin{pmatrix}4 & 1 & 2\\ -5 & 0 & 2\end{pmatrix},$$

find the matrix:

(a) $2A+3B-C$ (b) $3A-2B+C$
(c) $A-2B+3C$ (d) $2(A+B+C)$

7. Solve each matrix equation:

(a) $A+\begin{pmatrix}1 & 1\\ 2 & 3\end{pmatrix}=\begin{pmatrix}4 & 7\\ 1 & 5\end{pmatrix}$ (b) $\begin{pmatrix}3 & 1\\ 2 & 5\end{pmatrix}+B=\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}$

(c) $\begin{pmatrix}3 & 2\\ -1 & 4\end{pmatrix}+C=\begin{pmatrix}4 & 7\\ 1 & 3\end{pmatrix}$ (d) $D-\begin{pmatrix}2 & 3\\ 1 & 4\end{pmatrix}=\begin{pmatrix}3 & 1\\ 4 & 6\end{pmatrix}$

(e) $\begin{pmatrix}3 & 2 & 1\\ 4 & 0 & 6\end{pmatrix}=\begin{pmatrix}4 & 2 & 1\\ 3 & 1 & 4\end{pmatrix}-E$ (f) $F-\begin{pmatrix}2 & 1\\ 3 & 4\\ 5 & 7\end{pmatrix}=\begin{pmatrix}1 & 3\\ 4 & 2\\ 9 & 8\end{pmatrix}$

(g) $3X=\begin{pmatrix}6 & -3\\ 0 & 9\end{pmatrix}$ (h) $2Y+\begin{pmatrix}1 & 4\\ 5 & 7\end{pmatrix}=\begin{pmatrix}5 & 6\\ 1 & 9\end{pmatrix}$

(i) $2P-3\begin{pmatrix}1 & 1 & 2\\ 2 & -1 & 0\end{pmatrix}=\begin{pmatrix}7 & 5 & 1\\ 6 & -2 & 3\end{pmatrix}$

8. Write down the **transpose** of each matrix:

(a) $A=\begin{pmatrix}2 & 0\\ -1 & -4\\ 1 & 3\end{pmatrix}$ (b) $B=\begin{pmatrix}3 & 4 & 1\\ -7 & 9 & 5\end{pmatrix}$ (c) $C=\begin{pmatrix}3 & -6 & 0\\ -4 & 7 & 1\\ 7 & 3 & -2\end{pmatrix}$

9. (a) Show that the matrix $A=\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$ is **symmetric**.

(b) Show that the matrix $B=\begin{pmatrix}0 & 1 & -5\\ -1 & 0 & 2\\ 5 & -2 & 0\end{pmatrix}$ is **skew-symmetric**.

10. (a) Given the matrix $A=\begin{pmatrix}1 & -4 & 7\\ -2 & 3 & 2\\ 1 & 0 & -6\end{pmatrix}$, find the matrix:

(i) A' (the transpose of A) (ii) $A+A'$ (iii) $A-A'$

(b) Show that the matrix $A+A'$ is symmetric.

(c) Show that the matrix $A-A'$ is skew-symmetric.

[It can be shown that the properties in (b) and (c) are true for any square matrix A. That is, given any square matrix A, the matrix $A+A'$ is always symmetric and the matrix $A-A'$ is always skew-symmetric.]

ANSWERS

1. (a) A: 2×2 B: 2×3 C: 4×3 D: 3×1

(b) 1 (c) -1 (d) -1

(e)(i) 3 (ii) 0 (iii) 6 (iv) 1 (v) -4 (v) 0

2. (a) 2×2 (b) 3×1 (c) 2×3 (d) 3×4 (e) 1×3 (f) 3×3

3. (a) $x=2,\ y=12$ (b) $x=1,\ y=-2$ (c) $x=4,\ y=3$
(d) $x=1,\ y=2$ (e) $x=2,\ y=-3$

4. (a) $\begin{pmatrix}7\\4\end{pmatrix}$ (b) $\begin{pmatrix}5\\1\end{pmatrix}$ (c) $\begin{pmatrix}q\\p\end{pmatrix}$ (d) $\begin{pmatrix}2k+4\\7k+1\end{pmatrix}$

(e) $\begin{pmatrix}5&7\\4&8\\7&8\end{pmatrix}$ (f) $\begin{pmatrix}5x\\-2y\end{pmatrix}$ (g) $\begin{pmatrix}1&5\\4&6\\13&-3\end{pmatrix}$ (h) $\begin{pmatrix}5&5&1\\-1&1&10\end{pmatrix}$

(i) $\begin{pmatrix}6&1&5\\3&0&0\\-2&3&-4\end{pmatrix}$ (j) $\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$ (k) $\begin{pmatrix}2\\-1\end{pmatrix}$ (l) $\begin{pmatrix}6\\-8\end{pmatrix}$

(m) $\begin{pmatrix}3a\\3b\end{pmatrix}$ (n) $\begin{pmatrix}-3m\\-n\end{pmatrix}$ (o) $\begin{pmatrix}4&1\\-2&-5\\-5&4\end{pmatrix}$ (p) $\begin{pmatrix}-1&2\\1&-3\end{pmatrix}$

(q) $\begin{pmatrix}5&1\\-3&3\\-1&3\end{pmatrix}$ (r) $\begin{pmatrix}-3\\5\\2\end{pmatrix}$ (s) $\begin{pmatrix}-2&-5\\4&2\end{pmatrix}$ (t) $\begin{pmatrix}b\\-a\end{pmatrix}$

(u) $\begin{pmatrix}3\\0\\1\end{pmatrix}$ (v) $\begin{pmatrix}-4&11&-3\\2&-1&1\\-1&0&0\end{pmatrix}$ (w) $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ (x) $\begin{pmatrix}1\\-1\\-2\end{pmatrix}$

(y) $\begin{pmatrix}2&4&-3\\2&1&-4\\1&0&-4\end{pmatrix}$ (z) $\begin{pmatrix}-3&-3\\-5&-2\end{pmatrix}$

5. (a) $\begin{pmatrix}5&-20&-30\\15&-15&-5\end{pmatrix}$ (b) $\begin{pmatrix}3&-9&-3\\12&-6&0\\15&3&9\end{pmatrix}$ (c) $\begin{pmatrix}-6&0&2\\4&2&-4\\-6&2&2\end{pmatrix}$

(d) $\begin{pmatrix}6&11&-4\\-9&-4&-6\\13&3&9\end{pmatrix}$

6. (a) $\begin{pmatrix}-2&0&-5\\18&8&5\end{pmatrix}$ (b) $\begin{pmatrix}7&-4&4\\-5&-1&6\end{pmatrix}$ (c) $\begin{pmatrix}13&0&8\\-19&-3&6\end{pmatrix}$

(d) $\begin{pmatrix}10&2&2\\0&6&10\end{pmatrix}$

7. (a) $A=\begin{pmatrix}3&6\\-1&2\end{pmatrix}$ (b) $B=\begin{pmatrix}-2&1\\-1&-2\end{pmatrix}$ (c) $C=\begin{pmatrix}1&5\\2&-1\end{pmatrix}$

(d) $D=\begin{pmatrix}5&4\\5&10\end{pmatrix}$ (e) $E=\begin{pmatrix}1&0&0\\-1&1&-2\end{pmatrix}$ (f) $F=\begin{pmatrix}3&4\\7&6\\14&15\end{pmatrix}$

(g) $X=\begin{pmatrix}2&-1\\0&3\end{pmatrix}$ (h) $Y=\begin{pmatrix}2&1\\-2&1\end{pmatrix}$ (i) $P=\begin{pmatrix}5&4&3\frac{1}{2}\\6&-2\frac{1}{2}&1\frac{1}{2}\end{pmatrix}$

8. (a) $A'=\begin{pmatrix}2&-1&1\\0&-4&3\end{pmatrix}$ (b) $B'=\begin{pmatrix}3&-7\\4&9\\1&5\end{pmatrix}$ (c) $C'=\begin{pmatrix}3&-4&7\\-6&7&3\\0&1&-2\end{pmatrix}$

9. (a) $A'=\begin{pmatrix}2&-1\\-1&2\end{pmatrix}=A$, so A is symmetric

(b) $B'=\begin{pmatrix}0&-1&5\\1&0&-2\\-5&2&0\end{pmatrix}=-B$, so B is skew-symmetric

10. (a)(i) $\begin{pmatrix}1&-2&1\\-4&3&0\\7&2&-6\end{pmatrix}$ (ii) $\begin{pmatrix}2&-6&8\\-6&6&2\\8&2&-12\end{pmatrix}$ (iii) $\begin{pmatrix}0&-2&6\\2&0&2\\-6&-2&0\end{pmatrix}$

MATRICES 2

MATRIX MULTIPLICATION

1. Find each matrix product:

(a) $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 2 & -1 \end{pmatrix}$ (f) $\begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}$

(g) $\begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ (h) $\begin{pmatrix} 3 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \end{pmatrix}$ (i) $\begin{pmatrix} 1 & 3 & 2 \\ -2 & 0 & 1 \\ 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(j) $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 4 & 5 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ -2 & 2 \\ 1 & 0 \end{pmatrix}$ (k) $\begin{pmatrix} -1 & 2 \\ 0 & 3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 & -2 \\ 3 & 1 & 0 \end{pmatrix}$ (l) $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$

(m) $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ (n) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$ (o) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

(p) $\begin{pmatrix} 4 & -2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 0 & -3 \end{pmatrix}$ (q) $\begin{pmatrix} 3 & -2 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 1 & 2 \end{pmatrix}$ (r) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(s) $\begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ (t) $\begin{pmatrix} -1 & 0 & -2 \\ 3 & 2 & -1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 & -1 & 1 \\ 4 & -3 & 2 & -2 \\ 2 & 0 & 1 & 1 \end{pmatrix}$

(u) $\begin{pmatrix} -2 & 1 & 0 & 2 \\ 3 & -2 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 & 0 \\ -2 & 1 & -1 & 3 \\ 2 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ (v) $\begin{pmatrix} 1 & 2 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}$

(w) $\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ (x) $\begin{pmatrix} 4 & 1 & 5 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & 2 \\ 3 & 1 \end{pmatrix}$ (y) $\begin{pmatrix} 3 & 1 & 4 \\ 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} 4 & 1 & 2 \\ 1 & 3 & 1 \\ 7 & 4 & 3 \end{pmatrix}$

(z) $\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ 2 & 1 & -1 \\ 0 & 2 & 2 \end{pmatrix}$

2. Given the 2×2 matrices $A = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$, find the matrix products AB and BA.

3. Given the matrices

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & -2 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 0 & 3 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ -3 & 2 & -2 \end{pmatrix} \text{ and } D = \begin{pmatrix} 3 & 1 & 1 & -2 \\ 1 & 1 & 0 & -1 \\ 2 & -2 & 1 & 2 \end{pmatrix},$$

find the matrix:

(a) AB (b) AC (c) AD (d) BA (e) CB (f) CD

4. Given the 2×2 matrices $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 5 \\ 1 & 6 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & -1 \\ 8 & 3 \end{pmatrix}$, find the matrix:

(a) AB (b) BC (c) ABC

5. Given the 2×2 matrices $A = \begin{pmatrix} 3 & 1 \\ 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$, find the matrices A', B', AB and BA.

Hence show that:
(a) $(AB)' = B'A'$
(b) $(BA)' = A'B'$

[It can be shown that the properties in (a) and (b) are true in general for any matrices A and B for which the relevant matrix products exist.]

6. Given the 2×2 matrices $A = \begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$, show that $A(B + C) = AB + AC$.

[It can be shown that this property is true in general for any matrices A, B and C for which the relevant matrix products exist.]

7. Find each matrix product and simplify the entries:

(a) $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

(b) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$

(c) $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

8. Given the 2×2 matrix $A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$, find the matrices A^2 and A^3.

9. Given the 3×3 matrices

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 3 & 0 & -2 \\ 6 & 5 & 1 \end{pmatrix},\ B = \begin{pmatrix} 4 & 7 & 2 \\ 1 & -1 & 0 \\ 2 & 3 & -1 \end{pmatrix},\ C = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & -1 \\ -2 & 3 & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & -1 \\ 2 & 4 & 7 \end{pmatrix},$$

find the matrix:

(a) AB (b) CD (c) $AB + CD$
(d) D^2 (e) $AB + D^2$ (f) C^2
(g) C^3

10. Given the 3×3 matrices $R = \begin{pmatrix} 2 & -1 & 1 \\ -3 & 4 & -3 \\ -5 & 5 & -4 \end{pmatrix}$ and $S = \begin{pmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{pmatrix}$, show that $RS = O$ (the zero matrix) and find the matrix product SR.

11. The matrices P, Q and R are defined by

$$P = \begin{pmatrix} 2 & 5 \\ 3 & -4 \end{pmatrix},\ Q = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 \\ -10 \end{pmatrix},$$

where x and y are real numbers.

Given that $PQ = R$, find the values of x and y.

12. The 2×2 matrix A is given by $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$.

(a) Find the matrices A^2, A^3 and A^4.
(b) Conjecture expressions in terms of n for the entries of the matrix A^n, where n is a positive integer.

13. Let $A = \begin{pmatrix} k & 1-k \\ 0 & k \end{pmatrix}$, where k is a fixed real number.

(a) Find the matrices A^2, A^3 and A^4, expressing the entries of these matrices in terms of k as simply as possible.
(b) Conjecture expressions in terms of k and n for the entries of the matrix A^n, where n is a positive integer.

MATRICES 3

IDENTITY MATRICES

1. Given the matrix $A=\begin{pmatrix}1 & 2\\ -3 & 0\end{pmatrix}$, find the matrix $A^2+3A-5I$, where I is the 2×2 identity matrix.

2. Given the matrix $A=\begin{pmatrix}2 & -4\\ 3 & 2\end{pmatrix}$, show that $A^3=kI$ for some real number k, where I is the 2×2 identity matrix.

3. (a) Given the matrix $A=\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}$, show that $A^2=2A+5I$, where I is the 2×2 identity matrix.

(b) Hence express the matrix A^3 in the form $pA+qI$ for some integers p and q.

4. (a) Given the 2×2 matrix $A=\begin{pmatrix}3 & 4\\ 2 & 1\end{pmatrix}$, show that $A^2=4A+5I$, where I is the 2×2 identity matrix.

(b) Hence express the matrix A^3 in the form $pA+qI$ for some integers p and q.

5. (a) Given the 2×2 matrix $A=\begin{pmatrix}2 & 3\\ -1 & 2\end{pmatrix}$, show that $A^2=4A-7I$, where I is the 2×2 identity matrix.

(b) Hence express the matrix A^3 in the form $pA+qI$ for some integers p and q.

6. (a) Given the 2×2 matrix $A=\begin{pmatrix}3 & 2\\ 4 & 1\end{pmatrix}$, find the values of the integers p and q such that $A^2=pA+qI$, where I is the 2×2 identity matrix.

(b) Hence express each of the matrices A^3, A^4 and A^5 in the form $xA+yI$ for some integers x and y.

7. (a) Given the 2×2 matrix $C=\begin{pmatrix}2 & 5\\ -3 & 1\end{pmatrix}$, find the values of the integers p and q such that $C^2=pA+qI$, where I is the 2×2 identity matrix.

(b) Hence express C^4 in the form $xA+yI$ for some integers x and y.

8. The square matrix A is such that $A^2=4A-3I$, where I is the corresponding identity matrix.

Find integers p and q such that $A^4=pA+qI$.

9. Let $A=\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & -1 & -1\end{pmatrix}$ and $B=\begin{pmatrix}1 & 0 & 1\\ 4 & -2 & -2\\ -3 & 2 & 1\end{pmatrix}$.

Show that $AB=kI$ for some real number k, where I is the 3×3 identity matrix.

Hence obtain the matrix A^2B.

10. For any angle x, B_x is defined to be the 2×2 matrix

$$B_x=\begin{pmatrix}\cos x & \sin x\\ \sin x & -\cos x\end{pmatrix}.$$

Show that $(B_x)^2=I$, where I is the 2×2 identity matrix.

11. A matrix A is said to be **orthogonal** if $A'A=I$, where A' is the transpose of A and I is an identity matrix.

Show that the 2×2 matrix $A=\begin{pmatrix}0\cdot 8 & 0\cdot 6\\ -0\cdot 6 & 0\cdot 8\end{pmatrix}$ is orthogonal.

12. The 3×3 matrices A and B are defined by

$$A=\begin{pmatrix}1 & -1 & 3\\ 2 & -1 & 9\\ 4 & -8 & 1\end{pmatrix} \text{ and } B=\begin{pmatrix}71 & a & -6\\ 34 & b & -3\\ -12 & c & 1\end{pmatrix},$$

where a, b and c are constants.

(a) Find the matrix $B-3A$.

(b) (i) Verify that $AB=I$, where I is the 3×3 identity matrix, provided that

$$a-b+3c=0$$
$$2a-b+9c=1$$
$$4a-8b+c=0$$

(ii) Use Gaussian elimination to find the values of a, b and c for which $AB=I$.

ANSWERS

1. $A^2 + 3A - 5I = \begin{pmatrix} -7 & 8 \\ -12 & -11 \end{pmatrix}$

2. $A^3 = \begin{pmatrix} -64 & 0 \\ 0 & -64 \end{pmatrix} = -64I$

3. (b) $A^3 = 9A + 10I$

4. (b) $A^3 = 21A + 20I$

5. (b) $A^3 = 9A - 28I$

6. (a) $p = 4$, $q = 5$ (b) $A^3 = 21A + 20I$;
$A^4 = 104A + 105I$;
$A^5 = 521A + 520I$

7. (a) $p = 3$, $q = -17$ (b) $C^4 = -75A + 136I$

8. $p = 40$, $q = -39$

9. $AB = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2I$; $A^2B = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 6 \\ 2 & -2 & -2 \end{pmatrix}$

11. $A'A = \begin{pmatrix} 0\cdot8 & -0\cdot6 \\ 0\cdot6 & 0\cdot8 \end{pmatrix}\begin{pmatrix} 0\cdot8 & 0\cdot6 \\ -0\cdot6 & 0\cdot8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, hence the matrix A is orthogonal

ANSWERS

1. (a) $\begin{pmatrix}4\\11\end{pmatrix}$ (b) $\begin{pmatrix}-1\\2\end{pmatrix}$ (c) $\begin{pmatrix}-5\\-5\end{pmatrix}$ (d) $\begin{pmatrix}5\\2\end{pmatrix}$

(e) $\begin{pmatrix}6&-4\\0&1\end{pmatrix}$ (f) $\begin{pmatrix}0&-4\\1&-6\end{pmatrix}$ (g) $\begin{pmatrix}8\\9\end{pmatrix}$ (h) $\begin{pmatrix}2&3\\-2&4\end{pmatrix}$

(i) $\begin{pmatrix}2\\0\\8\end{pmatrix}$ (j) $\begin{pmatrix}-4&3\\-3&2\\-9&11\end{pmatrix}$ (k) $\begin{pmatrix}7&0&2\\9&3&0\\5&-3&4\end{pmatrix}$ (l) $\begin{pmatrix}-1&4\\-2&2\end{pmatrix}$

(m) $\begin{pmatrix}2&8\\-1&-1\end{pmatrix}$ (n) $\begin{pmatrix}6&2\\2&-2\end{pmatrix}$ (o) $\begin{pmatrix}1&5\\5&4\end{pmatrix}$ (p) $\begin{pmatrix}12&-2\\3&-8\end{pmatrix}$

(q) $\begin{pmatrix}10&-10\\-3&-6\end{pmatrix}$ (r) $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ (s) $\begin{pmatrix}4&5\\7&-7\end{pmatrix}$

(t) $\begin{pmatrix}-4&-2&-1&-3\\6&0&0&-2\\8&-4&3&-3\end{pmatrix}$ (u) $\begin{pmatrix}-4&-1&-5&5\\6&0&9&-5\\2&0&3&1\\-5&4&2&3\end{pmatrix}$

(v) $\begin{pmatrix}5&-2\\16&-4\end{pmatrix}$ (w) $\begin{pmatrix}-1&3\\1&7\end{pmatrix}$ (x) $\begin{pmatrix}22&19\\14&13\end{pmatrix}$ (y) $\begin{pmatrix}41&22&19\\41&27&19\end{pmatrix}$

(z) $\begin{pmatrix}2&-3&-1\\4&-2&-2\\0&2&-2\end{pmatrix}$

2. $AB=\begin{pmatrix}5&4\\-1&1\end{pmatrix}$; $BA=\begin{pmatrix}6&3\\-3&0\end{pmatrix}$

3. (a) $\begin{pmatrix}0&2\\7&7\end{pmatrix}$ (b) $\begin{pmatrix}7&-3&7\\-4&4&9\end{pmatrix}$ (c) $\begin{pmatrix}5&5&1&-7\\9&-1&4&-2\end{pmatrix}$

(d) $\begin{pmatrix}8&-3&1\\-1&-4&3\\9&-6&3\end{pmatrix}$ (e) $\begin{pmatrix}1&11\\4&0\\-7&-10\end{pmatrix}$ (f) $\begin{pmatrix}9&-5&4&4\\3&3&1&-5\\-11&3&-5&0\end{pmatrix}$

4. (a) $\begin{pmatrix}5&17\\1&6\end{pmatrix}$ (b) $\begin{pmatrix}46&12\\50&17\end{pmatrix}$ (c) $\begin{pmatrix}146&46\\50&17\end{pmatrix}$

5. $A'=\begin{pmatrix}3&4\\1&5\end{pmatrix}$; $B'=\begin{pmatrix}1&1\\-2&3\end{pmatrix}$; $AB=\begin{pmatrix}4&-3\\9&7\end{pmatrix}$; $BA=\begin{pmatrix}-5&-9\\15&16\end{pmatrix}$

(a) $(AB)'=\begin{pmatrix}4&9\\-3&7\end{pmatrix}$ and $B'A'=\begin{pmatrix}1&1\\-2&3\end{pmatrix}\begin{pmatrix}3&4\\1&5\end{pmatrix}=\begin{pmatrix}4&9\\-3&7\end{pmatrix}$, hence $(AB)'=B'A'$

(b) $(BA)'=\begin{pmatrix}-5&15\\-9&16\end{pmatrix}$ and $A'B'=\begin{pmatrix}3&4\\1&5\end{pmatrix}\begin{pmatrix}1&1\\-2&3\end{pmatrix}=\begin{pmatrix}-5&15\\-9&16\end{pmatrix}$, hence $(BA)'=A'B'$

6. $A(B+C)=\begin{pmatrix}1&5\\3&2\end{pmatrix}\begin{pmatrix}3&-1\\6&5\end{pmatrix}=\begin{pmatrix}33&24\\21&7\end{pmatrix}$

and $AB+AC=\begin{pmatrix}1&5\\3&2\end{pmatrix}\begin{pmatrix}2&0\\4&1\end{pmatrix}+\begin{pmatrix}1&5\\3&2\end{pmatrix}\begin{pmatrix}1&-1\\2&4\end{pmatrix}$
$=\begin{pmatrix}22&5\\14&2\end{pmatrix}+\begin{pmatrix}11&19\\7&5\end{pmatrix}$
$=\begin{pmatrix}33&24\\21&7\end{pmatrix}$,

hence $A(B+C)=AB+AC$

7. (a) $\begin{pmatrix}1\\0\end{pmatrix}$ (b) $\begin{pmatrix}0\\1\end{pmatrix}$ (c) $\begin{pmatrix}1&0\\0&1\end{pmatrix}$

8. $A^2=\begin{pmatrix}-1&4\\-2&-1\end{pmatrix}$; $A^3=\begin{pmatrix}-5&2\\-1&-5\end{pmatrix}$

9. (a) $\begin{pmatrix}20&29&7\\8&15&8\\31&40&11\end{pmatrix}$ (b) $\begin{pmatrix}13&14&20\\1&-4&-8\\5&-2&-3\end{pmatrix}$ (c) $\begin{pmatrix}33&43&27\\9&11&0\\36&38&8\end{pmatrix}$

(d) $\begin{pmatrix}7&2&-1\\4&-1&-7\\30&30&45\end{pmatrix}$ (e) $\begin{pmatrix}27&31&6\\12&14&1\\61&70&56\end{pmatrix}$ (f) $\begin{pmatrix}-2&12&5\\2&-2&-1\\-4&1&-9\end{pmatrix}$

(g) $\begin{pmatrix} -14 & 25 & -18 \\ 6 & -3 & 8 \\ 10 & -30 & -13 \end{pmatrix}$

10. $SR = O$ also

11. $x = -2,\ y = 1$

12. (a) $A^2 = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$; $A^3 = \begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$; $A^4 = \begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$

(b) $A^n = \begin{pmatrix} n+1 & n \\ -n & -n+1 \end{pmatrix}$

13. (a) $A^2 = \begin{pmatrix} k^2 & 2k(1-k) \\ 0 & k^2 \end{pmatrix}$; $A^3 = \begin{pmatrix} k^3 & 3k^2(1-k) \\ 0 & k^3 \end{pmatrix}$; $A^4 = \begin{pmatrix} k^4 & 4k^3(1-k) \\ 0 & k^4 \end{pmatrix}$

(b) $A^n = \begin{pmatrix} k^n & nk^{n-1}(1-k) \\ 0 & k^n \end{pmatrix}$

MATRICES 4

DETERMINANTS OF SQUARE MATRICES

1. Find the determinant of each 2×2 matrix:

(a) $\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 12 & 3 \\ 8 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 5 & -2 \\ -3 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 \\ 3 & 7 \end{pmatrix}$ (e) $\begin{pmatrix} -4 & -5 \\ 2 & -1 \end{pmatrix}$ (f) $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

(g) $\begin{pmatrix} 7 & 1 \\ -2 & -1 \end{pmatrix}$ (h) $\begin{pmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{pmatrix}$

2. Find the value or values of x in each case:

(a) $\begin{vmatrix} 2 & 1 \\ 1 & x \end{vmatrix} = 7$ (b) $\begin{vmatrix} 3x & 1 \\ 1 & x \end{vmatrix} = 47$ (c) $\begin{vmatrix} 3 & -1 \\ 2x & 4 \end{vmatrix} = 26$

(d) $\begin{vmatrix} 2x & x \\ -3 & x \end{vmatrix} = 2$ (e) $\begin{vmatrix} x & x \\ 3 & x \end{vmatrix} = 10$ (f) $\begin{vmatrix} e^x & e \\ 1 & e^x \end{vmatrix} = 0$

3. (a) Given the 2×2 matrix $P=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, find the determinant of the matrix $2P$.

(b) Given the 2×2 matrix $Q=\begin{pmatrix} 3 & 5 \\ 1 & 3 \end{pmatrix}$, find the determinant of the matrix $3Q$.

4. Find the determinant of each 3×3 matrix:

(a) $\begin{pmatrix} 2 & 1 & 3 \\ 4 & -2 & 5 \\ 3 & 6 & 8 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 2 & 1 \\ 3 & 4 & 5 \\ 2 & -7 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & 1 & 3 \\ 1 & 3 & 2 \\ 2 & -1 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 3 \\ -1 & -1 & -1 \end{pmatrix}$ (e) $\begin{pmatrix} -2 & 1 & 3 \\ 0 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 3 & -2 & 0 \\ 3 & -1 & 1 \\ -2 & 1 & 1 \end{pmatrix}$

(g) $\begin{pmatrix} -2 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{pmatrix}$ (h) $\begin{pmatrix} 2 & -1 & 2 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ (i) $\begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 4 \\ 1 & -2 & 3 \end{pmatrix}$

(j) $\begin{pmatrix} 3 & -1 & 4 \\ 2 & 1 & -3 \\ 5 & 1 & 6 \end{pmatrix}$ (k) $\begin{pmatrix} 1 & 1 & 4 \\ 1 & -2 & 2 \\ 0 & 3 & 1 \end{pmatrix}$

5. Find the value of λ for which the determinant of the matrix

$$A=\begin{pmatrix} 2 & 3 & -2 \\ 1 & \lambda & 4 \\ -3 & 1 & 2 \end{pmatrix}$$

is zero.

6. Find the value of λ for which the determinant of the matrix

$$A=\begin{pmatrix} 3 & 2 & -1 \\ 1 & \lambda & 1 \\ -2 & 3 & 2 \end{pmatrix}$$

is zero.

7. Find the value of x for which

$$\det\begin{pmatrix} 2 & 3 & -2 \\ x & 1 & 0 \\ 4 & -5 & 3 \end{pmatrix} = 0.$$

8. The 3×3 matrix A is given by $A=\begin{pmatrix} k & 2 & 3 \\ 1 & 2 & -1 \\ 2 & -3 & k \end{pmatrix}$.

Find the values of k for which the determinant of the matrix A is zero.

9. Find the values of x for which

$$\det\begin{pmatrix} 1 & -3 & 2 \\ 4 & x & 5 \\ 2 & -1 & x+1 \end{pmatrix} = 1.$$

10. Let $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$.

(a) Write down the matrix $A - \lambda I$, where $\lambda \in \mathbf{R}$ and I is the 3×3 identity matrix.

(b) Find the values of λ for which the determinant of $A - \lambda I$ is zero.

11. Given that

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 1 \\ 0 & 0 & k \end{pmatrix},$$

obtain the matrix $A^2 - 2I$ and find the values of k for which the determinant of $A^2 - 2I$ is zero. (I is the 3×3 identity matrix.)

12. The 3×3 matrix A is given by

$$A = \begin{pmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{pmatrix}.$$

It is required to find all the values of the real number λ for which the determinant of the matrix $A - \lambda I$ is zero, where I is the 3×3 identity matrix.

(a) Write down the matrix $A - \lambda I$, where λ is a real number.

(b) Show that the required values of λ satisfy the cubic equation

$$\lambda^3 - \lambda^2 - 21\lambda + 45 = 0$$

and solve this equation to find all the required values of λ.

[Note: Given a square matrix A, the values of λ for which the determinant of the matrix $A - \lambda I$ is zero are known as the **eigenvalues** of the matrix A.]

ANSWERS

1. (a) 14 (b) 0 (c) −11 (d) 7 (e) 14 (f) −1
(g) −5 (h) 1

2. (a) $x = 4$ (b) $x = \pm 4$ (c) $x = 7$
(d) $x = \frac{1}{2}$ or $x = -2$ (e) $x = 5$ or $x = -2$ (f) $x = \frac{1}{2}$

3. (a) −8 (b) 36

4. (a) −19 (b) −74 (c) 2 (d) 0 (e) 2 (f) 4
(g) 20 (h) −14 (i) 23 (j) 66 (k) 3

5. $\lambda = -26$

6. $\lambda = 5$

7. $x = -14$

8. $k = 5$ or $k = -\frac{5}{2}$

9. $x = 2$ or $x = -11$

10. (a) $A - \lambda I = \begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & -\lambda & -1 \\ -1 & 1 & -\lambda \end{pmatrix}$ (b) $\lambda = 0$ or $\lambda = 1$

11. $A^2 - 2I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & k^2-2 & 2k \\ 0 & 0 & k^2-2 \end{pmatrix}$; $k = \pm\sqrt{2}$

12. (a) $A - \lambda I = \begin{pmatrix} 2-\lambda & -2 & 3 \\ -2 & -1-\lambda & 6 \\ 1 & 2 & -\lambda \end{pmatrix}$ (b) $\lambda = 3$ or $\lambda = -5$

MATRICES 5

INVERSE MATRICES

1. Let $A = \begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}$.

(a) Calculate the determinant of A and explain why the matrix A is invertible.

(b) Show that $A^2 = 10A - I$, where I is the 2×2 identity matrix.

(c) Hence show that $A^{-1} = 10I - A$ and obtain the matrix A^{-1}.

2. Let $A = \begin{pmatrix} 2 & 3 \\ 7 & 11 \end{pmatrix}$.

(a) Show that the matrix A is invertible.

(b) Show that $A^2 = 13A - I$, where I is the 2×2 identity matrix.

(c) Hence show that $A^{-1} = 13I - A$ and obtain the matrix A^{-1}.

3. Let A be the matrix $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.

(a) Show that A is invertible.

(b) Show that $A^2 - 7A = kI$ for some real number k, where I is the 2×2 identity matrix.

(c) Hence obtain the inverse matrix A^{-1}.

4. The 2×2 matrix A is given by $\begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix}$.

(a) Show that A is invertible.

(b) Find the values of the integers p and q for which $A^2 = pA + qI$, where I is the 2×2 identity matrix.

(c) Hence obtain the inverse matrix A^{-1}.

5. Let $C = \begin{pmatrix} 2 & 5 \\ -3 & 1 \end{pmatrix}$.

(a) Show that the matrix C is invertible.

(b) By expressing the matrix C^2 in the form $C^2 = pC + qI$ for integers p and q, or otherwise, obtain the inverse matrix C^{-1}.

6. Find the value or values of k for which each 2×2 matrix is singular (non-invertible):

(a) $\begin{pmatrix} 2 & 2 \\ 4 & k \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2k \\ 3 & 6 \end{pmatrix}$ (c) $\begin{pmatrix} 1-k & -1 \\ 3 & 1+k \end{pmatrix}$

(d) $\begin{pmatrix} 2+k & -6 \\ 4 & 3-k \end{pmatrix}$

7. Let $A = \begin{pmatrix} 2 & 5 \\ 4 & 1 \end{pmatrix}$.

(a) Write down the matrix $A - \lambda I$, where λ is a real number.

(b) Find the values of λ for which the matrix $A - \lambda I$ is singular (non-invertible).

8. Let $A = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}$.

(a) Write down the matrix $A - \lambda I$, where λ is a real number.

(b) Find the values of λ for which the matrix $A - \lambda I$ is singular (non-invertible)

9. The 3×3 matrices A and B are given by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix}.$$

Show that $AB = kI$ for some real number k, where I is the 3×3 identity matrix.

Hence obtain the inverse matrix A^{-1}.

10. The 3×3 matrices A and B are given by

$$A = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}.$$

(a) Calculate the determinant of A and explain why the matrix A is invertible.

(b) By considering the matrix product AB, obtain the inverse matrix A^{-1}.

11. The 3×3 matrix A is given by

$$A = \begin{pmatrix} 2 & -1 & 4 \\ 4 & 0 & 2 \\ 3 & -2 & 7 \end{pmatrix}.$$

Find the matrices A^2 and A^3 and verify that $A^3 = 9A^2 - 10A - 2I$, where I is the 3×3 identity matrix.

Hence obtain the inverse matrix A^{-1}.

12. Let $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$.

(a) Show that the matrix A is invertible.

(b) Use row operations to find the inverse matrix A^{-1}.

(c) Verify that $AA^{-1} = I$ and $A^{-1}A = I$, where I is the 2×2 identity matrix.

13. Use row operations to find the inverse of each 2×2 matrix (you may assume without proof that each matrix is invertible):

(a) $\begin{pmatrix} 5 & 4 \\ 4 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 9 & 2 \\ 13 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 4 & -11 \\ -1 & 3 \end{pmatrix}$

(e) $\begin{pmatrix} -3 & 4 \\ 8 & -11 \end{pmatrix}$ (f) $\begin{pmatrix} 3 & 1 \\ 7 & 3 \end{pmatrix}$ (g) $\begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}$ (h) $\begin{pmatrix} -2 & -6 \\ 0 & -2 \end{pmatrix}$

(i) $\begin{pmatrix} 5 & -4 \\ -3 & 3 \end{pmatrix}$ (j) $\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$ (k) $\begin{pmatrix} 2 & 5 \\ -2 & -4 \end{pmatrix}$

14. Use row operations to find the inverse of each 3×3 matrix (you may assume without proof that each matrix is invertible):

(a) $\begin{pmatrix} 1 & 0 & 2 \\ 4 & 1 & 1 \\ 3 & 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -1 & 1 \\ -2 & 2 & -3 \\ 4 & -5 & 6 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 2 & -2 \\ 2 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}$ (h) $\begin{pmatrix} 5 & 10 & 2 \\ 1 & 5 & 2 \\ 4 & 3 & -1 \end{pmatrix}$

ANSWERS

(1) (a) $\det A = 1$; $\det A \neq 0$, so the matrix A is invertible

(c) $A^{-1} = \begin{pmatrix} 5 & -4 \\ -6 & 5 \end{pmatrix}$

(2) (a) $\det A = 1$; $\det A \neq 0$, so the matrix A is invertible

(c) $A^{-1} = \begin{pmatrix} 11 & -3 \\ -7 & 2 \end{pmatrix}$

(3) (a) $\det A = 10$; $\det A \neq 0$, so the matrix A is invertible

(b) $A^2 - 7A = -10I$

(c) $A^{-1} = \begin{pmatrix} \frac{3}{10} & -\frac{1}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix} = \begin{pmatrix} 0.3 & -0.1 \\ -0.2 & 0.4 \end{pmatrix}$

(4) (a) $\det A = 2$; $\det A \neq 0$, so the matrix A is invertible

(b) $p = 5$, $q = -2$

(c) $A^{-1} = \begin{pmatrix} 1\frac{1}{2} & 2 \\ \frac{1}{2} & 1 \end{pmatrix}$

(5) (a) $\det C = 17$; $\det C \neq 0$, so the matrix C is invertible

(b) $C^{-1} = \begin{pmatrix} \frac{1}{17} & -\frac{5}{17} \\ \frac{3}{17} & \frac{2}{17} \end{pmatrix}$

(6) (a) $k = 4$ (b) $k = 1$ (c) $k = \pm 2$

(d) $k = 6$ or $k = -5$

(7) (a) $A - \lambda I = \begin{pmatrix} 2-\lambda & 5 \\ 4 & 1-\lambda \end{pmatrix}$ (b) $\lambda = 6$ or $\lambda = -3$

(8) (a) $A - \lambda I = \begin{pmatrix} 3-\lambda & 5 \\ 4 & 2-\lambda \end{pmatrix}$ (b) $\lambda = 7$ or $\lambda = -2$

(9) $AB = 2I$; $A^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 2 & -1 & -1 \\ -1\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}$

(10) (a) $\det A = 36$; $\det A \neq 0$, so the matrix A is invertible

(b) $AB = 6I$; $A^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{3} & 1\frac{1}{6} \end{pmatrix}$

(11) $A^2 = \begin{pmatrix} 12 & -10 & 34 \\ 14 & -8 & 30 \\ 19 & -17 & 57 \end{pmatrix}$; $A^3 = \begin{pmatrix} 86 & -80 & 266 \\ 86 & -74 & 250 \\ 141 & -133 & 441 \end{pmatrix}$;

$A^{-1} = \begin{pmatrix} -2 & \frac{1}{2} & 1 \\ 11 & -1 & -6 \\ 4 & -\frac{1}{2} & -2 \end{pmatrix}$

(12) (a) $\det A = 1$; $\det A \neq 0$, so the matrix A is invertible

(b) $A^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

(13) (a) $\begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & -2 \\ -13 & 9 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 11 \\ 1 & 4 \end{pmatrix}$ (e) $\begin{pmatrix} -11 & -4 \\ -8 & -3 \end{pmatrix}$ (f) $\begin{pmatrix} 1\frac{1}{2} & -\frac{1}{2} \\ -3\frac{1}{2} & 1\frac{1}{2} \end{pmatrix}$

(g) $\begin{pmatrix} -3 & 5 \\ 2 & -3 \end{pmatrix}$ (h) $\begin{pmatrix} -\frac{1}{2} & 1\frac{1}{2} \\ 0 & -\frac{1}{2} \end{pmatrix}$ (i) $\begin{pmatrix} 1 & 1\frac{1}{3} \\ 1 & 1\frac{2}{3} \end{pmatrix}$

(j) $\begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$ (k) $\begin{pmatrix} -2 & -2\frac{1}{2} \\ 1 & 1 \end{pmatrix}$

(14) (a) $\begin{pmatrix} -1 & 2 & -2 \\ 3 & -6 & 7 \\ 1 & -1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -1 & -1 \\ 0 & -2 & -1 \\ -2 & -1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} -1 & -6 & 4 \\ 2 & 11 & -7 \\ 0 & 2 & -1 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & -3 & 7 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{pmatrix}$ (h) $\begin{pmatrix} -11 & 16 & 10 \\ 9 & -13 & -8 \\ -17 & 25 & 15 \end{pmatrix}$