DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

1. Use the chain rule to differentiate each function with respect to x and simplify where possible.

(a) $y = \sin^{-1}(2x)$ (b) $y = \cos^{-1}(3x)$ (c) $y = \tan^{-1}(4x)$

(d) $y = \sin^{-1}(x^2)$ (e) $y = \tan^{-1}(x^3)$ (f) $y = \cos^{-1}(6x)$

(g) $y = \tan^{-1}(x+2)$ (h) $y = \sin^{-1}(\sqrt{x})$ (i) $y = \tan^{-1}(\sqrt{x})$

(j) $y = \tan^{-1}(e^x)$ (k) $y = \sin^{-1}(x-1)$ (l) $y = \tan^{-1}(2x+1)$

(m) $y = \tan^{-1}(\sqrt{x-1})$ (n) $y = \cos^{-1}(2x-1)$ (o) $y = \sin^{-1}(e^x)$

(p) $y = \tan^{-1}(e^{2x})$ (q) $y = \sin^{-1}\left(\frac{x}{2}\right)$ (r) $y = \sin^{-1}\left(\frac{1}{x}\right)$

(s) $y = \tan^{-1}(\sqrt{2x-1})$ (t) $y = \sin^{-1}(\cos x)$

2. Given $y = \sin^{-1}(\sin x)$, show that $\frac{dy}{dx} = 1$ and explain this answer.

3. (a) Show that $\frac{d}{dx}\left(\frac{x-1}{x+1}\right) = \frac{2}{(x+1)^2}$.

(b) Hence, given $y = \tan^{-1}\left(\frac{x-1}{x+1}\right)$, show that $\frac{dy}{dx} = \frac{1}{x^2+1}$.

4. (a) Show that $\frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{(x-1)e^x}{x^2}$.

(b) Hence, given $y = \tan^{-1}\left(\frac{e^x}{x}\right)$, show that $\frac{dy}{dx} = \frac{(x-1)e^x}{x^2+e^{2x}}$.

5. (a) Show that $\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1-\ln x}{x^2}$.

(b) Hence, given $y = \tan^{-1}\left(\frac{\ln x}{x}\right)$, show that $\frac{dy}{dx} = \frac{1-\ln x}{x^2+(\ln x)^2}$.

6. Use the product rule to differentiate each function with respect to x.

(a) $y = x\sin^{-1} x$ (b) $y = x\tan^{-1} x$ (c) $y = x^2\cos^{-1} x$

(d) $y = x^3\sin^{-1} x$ (e) $y = (1+x^2)\tan^{-1} x$ (f) $y = e^x\cos^{-1} x$

(g) $y = \ln x.\tan^{-1} x$ (h) $y = e^{2x}\sin^{-1} x$ (i) $y = \sqrt{1-x^2}\sin^{-1} x$

ANSWERS

1. (a) $\dfrac{2}{\sqrt{1-4x^2}}$ (b) $-\dfrac{3}{\sqrt{1-9x^2}}$ (c) $\dfrac{4}{1+16x^2}$ (d) $\dfrac{2x}{\sqrt{1-x^4}}$

(e) $\dfrac{3x^2}{1+x^6}$ (f) $-\dfrac{6}{\sqrt{1-36x^2}}$ (g) $\dfrac{1}{x^2+4x+5}$ (h) $\dfrac{1}{2\sqrt{x-x^2}}$

(i) $\dfrac{1}{2\sqrt{x}(1+x)}$ (j) $\dfrac{e^x}{1+e^{2x}}$ (k) $\dfrac{1}{\sqrt{2x-x^2}}$ (l) $\dfrac{1}{2x^2+2x+1}$

(m) $\dfrac{1}{2x\sqrt{x-1}}$ (n) $-\dfrac{1}{\sqrt{x-x^2}}$ (o) $\dfrac{e^x}{\sqrt{1-e^{2x}}}$ (p) $\dfrac{2e^{2x}}{1+e^{4x}}$

(q) $\dfrac{1}{\sqrt{4-x^2}}$ (r) $-\dfrac{1}{x\sqrt{x^2-1}}$ (s) $\dfrac{1}{2x\sqrt{2x-1}}$ (t) -1

2. $y=\sin^{-1}(\sin x)=x$, so $\dfrac{dy}{dx}=1$.

6. (a) $\dfrac{x}{\sqrt{1-x^2}}+\sin^{-1}x$ (b) $\dfrac{x}{1+x^2}+\tan^{-1}x$ (c) $2x\cos^{-1}x-\dfrac{x^2}{\sqrt{1-x^2}}$

(d) $\dfrac{x^3}{\sqrt{1-x^2}}+3x^2\sin^{-1}x$ (e) $1+2x\tan^{-1}x$ (f) $e^x\cos^{-1}x-\dfrac{e^x}{\sqrt{1-x^2}}$

(g) $\dfrac{\ln x}{1+x^2}+\dfrac{\tan^{-1}x}{x}$ (h) $\dfrac{e^{2x}}{\sqrt{1-x^2}}+2e^{2x}\sin^{-1}x$ (i) $1-\dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}$

IMPLICIT DIFFERENTIATION

1. y is a function of x defined implicitly by each equation below.
   Find an expression for $\frac{dy}{dx}$ in terms of x and y in each case.

(a) $x^2 + y^2 = 1$ (b) $x^2 - y^2 = 4$

(c) $x^2 + 2y^2 = 9$ (d) $x^2 - \ln y = 0$

(e) $x^2 + e^y = 1$ (f) $\sin x + e^y = 0$

(g) $y^2 = 2x + 2y$ (h) $\sin x + \sin y = 0$

(i) $y^2 - 3x = 4x^2$ (j) $2y^2 - x^2 = 1$

(k) $3x^2 + y^2 = 5x$ (l) $3y = 2x^3 + \cos y$

(m) $\ln(x + y) = x$ (n) $\sin x + 2\cos y = 1$

(o) $x^4 = x^2 - y^2$ (p) $2x^2 + 2y^2 - 4x + 4y - 9 = 0$

2. y is a function of x defined implicitly by each equation below.
   Find an expression for $\frac{dy}{dx}$ in terms of x and y in each case.

(a) $xy = 4$ (b) $x^2 + xy = 1$

(c) $x^3 + 1 = xy$ (d) $xy = x^2 + y^2$

(e) $x^2 + xy + y^2 = 0$ (f) $x^2 + 4xy + y^2 = 8$

(g) $x^2 - xy + 3y^2 = 10$ (h) $xy^2 = 1$

(i) $xe^y = 1$ (j) $x^2y = 9$

(k) $xy = x - y$ (l) $xy + y^2 = 2$

(m) $y^3 + 2xy = x^2 + 1$ (n) $x^3 + xy = 1 - y$

(o) $x \ln y = x^2 + y^2$ (p) $x \sin y + y = \cos x$

(q) $xy^2 + 1 = y$ (r) $x^3 - xy + y^2 = 1$

3. y is defined implicitly in terms of x by the equation $x^2 + y^2 = 25$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.
(b) Find the equation of the tangent at the point (3, –4) on the curve.

4. y is defined implicitly in terms of x by the equation $x^2 - y^2 = 9$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.
(b) Find the equation of the tangent at the point (5, 4) on the curve.

5. y is defined implicitly in terms of x by the equation $3x^2 + 5y^2 = 17$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.
(c) Find the equation of the tangent at the point (–2, 1) on the curve.

6. y is defined implicitly in terms of x by the equation $xy = 4$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.
(b) Find the equation of the tangent at the point (1, 4) on the curve.

7. y is defined implicitly in terms of x by the equation $x^2y = 9$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.
(b) Find the equation of the tangent at the point (3, 1) on the curve.

8. y is defined implicitly in terms of x by the equation $x^2 + 4xy - y^2 = 16$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.
(b) Find the equation of the tangent at the point (2, 2) on the curve.

9. y is defined implicitly in terms of x by the equation $x^2 - 4x + y^2 - 6y = 12$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.
(b) Find the equation of the tangent at the point (5, 7) on the curve.

10. y is defined implicitly in terms of x by the equation $x^2y + xy^2 = 2$.

(d) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

(e) Find the equation of the tangent at the point (1, 1) on the curve.

11. y is defined implicitly in terms of x by the equation $y^2 + 2xy = x^3 + 7$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

(b) Find the equation of the tangent at the point (1, –4) on the curve.

12. y is defined implicitly in terms of x by the equation $xy + y^2 = 2$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

(b) Find the equation of the tangent at the point (1, 1) on the curve.

13. y is defined implicitly in terms of x by the equation $y^3 + 3xy = 3x^2 + 2$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

(b) Find the equation of the tangent at the point (2, 1) on the curve.

14. y is defined implicitly in terms of x by the equation $x^3 - 2y^3 = 3xy$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

(b) Find the equation of the tangent at the point (2, 1) on the curve.

15. y is defined implicitly in terms of x by the equation $x^2 + y^2 = 4$.

Show that : (a) $\frac{dy}{dx} = -\frac{x}{y}$ (b) $\frac{d^2y}{dx^2} = -\frac{(x^2 + y^2)}{y^3}$

16. y is defined implicitly in terms of x by the equation $x^2 + xy = 3$.

Show that : (a) $\frac{dy}{dx} = -\frac{(2x + y)}{x}$ (b) $\frac{d^2y}{dx^2} = \frac{2(x + y)}{x^2}$

17. y is defined implicitly in terms of x by the equation $x^2 - y^2 = 4$.

Show that : (a) $\frac{dy}{dx} = \frac{x}{y}$ (b) $\frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}$

18. y is defined implicitly in terms of x by the equation $y^2 - x^2 = 2x$.

Show that : (a) $\frac{dy}{dx} = \frac{1 + x}{y}$ (b) $\frac{d^2y}{dx^2} = \frac{y^2 - (1 + x)^2}{y^3}$

19. y is defined implicitly in terms of x by the equation $xy = e^x$.

Show that : (a) $\frac{dy}{dx} = \frac{e^x - y}{x}$ (b) $\frac{d^2y}{dx^2} = \frac{(x - 2)e^x + 2y}{x^2}$

20. y is defined implicitly in terms of x by the equation $xy + y^2 = 1$.

Show that : (a) $\frac{dy}{dx} = -\frac{y}{(x + 2y)}$ (b) $\frac{d^2y}{dx^2} = \frac{2y(x + y)}{(x + 2y)^3}$

ANSWERS

1. (a) $\frac{dy}{dx}=-\frac{x}{y}$ (b) $\frac{dy}{dx}=\frac{x}{y}$ (c) $\frac{dy}{dx}=-\frac{x}{2y}$

(d) $\frac{dy}{dx}=2xy$ (e) $\frac{dy}{dx}=-\frac{2x}{e^y}$ (f) $\frac{dy}{dx}=-\frac{\cos x}{e^y}$

(g) $\frac{dy}{dx}=\frac{1}{y-1}$ (h) $\frac{dy}{dx}=-\frac{\cos x}{\cos y}$ (i) $\frac{dy}{dx}=\frac{8x+3}{2y}$

(j) $\frac{dy}{dx}=\frac{x}{2y}$ (k) $\frac{dy}{dx}=\frac{5-6x}{2y}$ (l) $\frac{dy}{dx}=\frac{6x^2}{3+\sin y}$

(m) $\frac{dy}{dx}=x+y-1$ (n) $\frac{dy}{dx}=\frac{\cos x}{2\sin y}$ (o) $\frac{dy}{dx}=\frac{x(1-2x^2)}{y}$

(p) $\frac{dy}{dx}=\frac{1-x}{1+y}$

2. (a) $\frac{dy}{dx}=-\frac{y}{x}$ (b) $\frac{dy}{dx}=-\frac{(2x+y)}{x}$ (c) $\frac{dy}{dx}=\frac{3x^2-y}{x}$

(d) $\frac{dy}{dx}=\frac{2x-y}{x-2y}$ (e) $\frac{dy}{dx}=-\frac{(2x+y)}{(x+2y)}$ (f) $\frac{dy}{dx}=-\frac{(x+2y)}{(2x+y)}$

(g) $\frac{dy}{dx}=\frac{y-2x}{6y-x}$ (h) $\frac{dy}{dx}=-\frac{y}{2x}$ (i) $\frac{dy}{dx}=-\frac{1}{x}$

(j) $\frac{dy}{dx}=-\frac{2y}{x}$ (k) $\frac{dy}{dx}=\frac{1-y}{1+x}$ (l) $\frac{dy}{dx}=-\frac{y}{(x+2y)}$

(m) $\frac{dy}{dx}=\frac{2(x-y)}{2x+3y^2}$ (n) $\frac{dy}{dx}=-\frac{(3x^2+y)}{(x+1)}$ (o) $\frac{dy}{dx}=\frac{y(2x-\ln y)}{x-2y^2}$

(p) $\frac{dy}{dx}=-\frac{(\sin x+\sin y)}{(x\cos y+1)}$ (q) $\frac{dy}{dx}=-\frac{y^2}{(2xy-1)}=\frac{y^2}{1-2xy}$

(r) $\frac{dy}{dx}=\frac{y-3x^2}{2y-x}$

3. (a) $\frac{dy}{dx}=-\frac{x}{y}$ (b) $4y=3x-25$

4. (a) $\frac{dy}{dx}=\frac{x}{y}$ (b) $4y=5x-9$

5. (a) $\frac{dy}{dx}=-\frac{3x}{5y}$ (b) $5y=6x+17$

6. (a) $\frac{dy}{dx}=-\frac{y}{x}$ (b) $y=-4x+8$

7. (a) $\frac{dy}{dx}=-\frac{2y}{x}$ (b) $3y=-2x+9$

8. (a) $\frac{dy}{dx}=-\frac{(x+2y)}{(2x-y)}=\frac{2y+x}{y-2x}$ (b) $y=-3x+8$

9. (a) $\frac{dy}{dx}=\frac{2-x}{y-3}$ (b) $4y=-3x+43$

10. (a) $\frac{dy}{dx}=-\frac{y(2x+y)}{x(x+2y)}$ (b) $y=-x+2$

11. (a) $\frac{dy}{dx}=\frac{3x^2-2y}{2(x+y)}$ (b) $6y=-11x-13$

12. (a) $\frac{dy}{dx}=-\frac{y}{(x+2y)}$ (b) $3y=-x+4$

13. (a) $\frac{dy}{dx}=\frac{2x-y}{x+y^2}$ (b) $y=x-1$

14. (a) $\frac{dy}{dx}=\frac{x^2-y}{x+2y^2}$ (b) $4y=3x-2$

LOGARITHMIC DIFFERENTIATION

1. Use logarithmic differentiation to find an expression for $\frac{dy}{dx}$ in each case.

(a)	$y = 3^x$	(b)	$y = 4^{2x}$	(c)	$y = 10^{3x}$
(d)	$y = x^2(x+1)^4$	(e)	$y = \frac{x^4}{(x+2)^6}$	(f)	$y = x^2\sqrt{2x+1}$
(g)	$y = x^6(2x-1)^3$	(h)	$y = \frac{2x+1}{\sqrt{2x+3}}$	(i)	$y = \frac{\sqrt{4x+1}}{x^2}$
(j)	$y = \frac{(2x+3)^2}{\sqrt{x+1}}$	(k)	$y = \pi^{4x}$	(l)	$y = \frac{x+1}{\sqrt{2x+1}}$
(m)	$y = \frac{x^3}{\sqrt{x+4}}$	(n)	$y = \frac{x^4(x+1)^3}{(x+2)^6}$	(o)	$y = \frac{x\sqrt{2x+1}}{x+1}$

2. Given $y = \frac{2^x}{2x+1}$, use logarithmic differentiation to show that $\frac{dy}{dx} = \frac{2^x}{2x+1}\left(\ln 2 - \frac{2}{2x+1}\right)$.

3. Given $y = (2x+1)^4(2x+3)^6$, use logarithmic differentiation to show that $\frac{dy}{dx} = 4(2x+1)^4(2x+3)^6\left(\frac{2}{2x+1} + \frac{3}{2x+3}\right)$.

4. Given $y = \frac{5^x}{x}$, use logarithmic differentiation to show that $\frac{dy}{dx} = \frac{5^x}{x}\left(\ln 5 - \frac{1}{x}\right)$.

5. Given $y = \sqrt{\frac{x+1}{x-1}}$, use logarithmic differentiation to show that $\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{x+1}{x-1}}\left(\frac{1}{x+1} - \frac{1}{x-1}\right)$.

6. Given $y = \frac{x^2(x^2+1)}{x^2+2}$, use logarithmic differentiation to show that $\frac{dy}{dx} = \frac{2x^2(x^2+1)}{x^2+2}\left(\frac{1}{x} + \frac{x}{x^2+1} - \frac{x}{x^2+2}\right)$.

7. Given $y = \frac{x\sqrt{x+3}}{x+2}$, use logarithmic differentiation to show that $\frac{dy}{dx} = \frac{x\sqrt{x+3}}{x+2}\left\{\frac{1}{x} + \frac{1}{2(x+3)} - \frac{1}{x+2}\right\}$.

8. Given $y = \frac{xe^{x^2}}{x+1}$, use logarithmic differentiation to show that $\frac{dy}{dx} = \frac{xe^{x^2}}{x+1}\left(\frac{1}{x} + 2x - \frac{1}{x+1}\right)$.

9. Given $y = \frac{\sqrt{x}(1-x)}{(3x+1)^{\frac{2}{3}}}$, use logarithmic differentiation to show that $\frac{dy}{dx} = \frac{\sqrt{x}(1-x)}{(3x+1)^{\frac{2}{3}}}\left(\frac{1}{2x} - \frac{1}{1-x} - \frac{2}{3x+1}\right)$.

10. Given $y = x^{\sin x}$, use logarithmic differentiation to show that $\frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \ln x \cos x\right)$.

11. Given $y = \frac{(2x+1)^2}{(2x+5)^3}$, use logarithmic differentiation to show that $\frac{dy}{dx} = \frac{2(2x+1)^2}{(2x+5)^3}\left(\frac{2}{2x+1} - \frac{3}{2x+5}\right)$.

12. Given $y = xe^{-x}\cos x$, use logarithmic differentiation to show that $\frac{dy}{dx} = xe^{-x}\cos x\left(\frac{1}{x} - 1 - \tan x\right)$.

13. Given $y = (\sin x)^x$, use logarithmic differentiation to show that $\frac{dy}{dx} = (\sin x)^x\{x\cot x + \ln(\sin x)\}$.

14. Given $y = (x^2 + 1)^x$, use logarithmic differentiation to show that $\frac{dy}{dx} = (x^2+1)^x\left\{\frac{2x^2}{x^2+1} + \ln(x^2+1)\right\}$.

15. $y = \frac{e^x \sin x}{x}$, use logarithmic differentiation to show that $\frac{dy}{dx} = \frac{e^x \sin x}{x}\left(1 + \cot x - \frac{1}{x}\right)$.

16. Given $y = x^{x^2}$, use logarithmic differentiation to show that $\frac{dy}{dx} = x^{x^2+1}(1 + 2\ln x)$.

17. Given $y = \frac{\sqrt{\sin x}}{x}$, use logarithmic differentiation to show that $\frac{dy}{dx} = \frac{\sqrt{\sin x}}{x}\left(\frac{1}{2}\cot x - \frac{1}{x}\right)$.

ANSWERS

1. (a) $\ln 3 \cdot 3^x$ (b) $2\ln 4 \cdot 4^{2x}$ (c) $3\ln 10 \cdot 10^{3x}$

(d) $2x^2(x+1)^4\left(\frac{1}{x} + \frac{2}{x+1}\right)$ (e) $\frac{2x^4}{(x+2)^6}\left(\frac{2}{x} - \frac{3}{x+2}\right)$

(f) $x^2\sqrt{2x+1}\left(\frac{2}{x} + \frac{1}{2x+1}\right)$ (g) $6x^6(2x-1)^3\left(\frac{1}{x} + \frac{1}{2x-1}\right)$

(h) $\frac{2x+1}{\sqrt{2x+3}}\left(\frac{2}{2x+1} - \frac{1}{2x+3}\right)$ (i) $\frac{2\sqrt{4x+1}}{x^2}\left(\frac{1}{4x+1} - \frac{1}{x}\right)$

(j) $\frac{(2x+3)^2}{\sqrt{x+1}}\left\{\frac{4}{2x+3} - \frac{1}{2(x+1)}\right\}$ (k) $4\ln\pi \cdot \pi^{4x}$

(l) $\frac{x+1}{\sqrt{2x+1}}\left(\frac{1}{x+1} - \frac{1}{2x+1}\right)$ (m) $\frac{x^3}{\sqrt{x+4}}\left\{\frac{3}{x} - \frac{1}{2(x+4)}\right\}$

(n) $\frac{x^4(x+1)^3}{(x+2)^6}\left(\frac{4}{x} + \frac{3}{x+1} - \frac{6}{x+2}\right)$ (o) $\frac{x\sqrt{2x+1}}{x+1}\left(\frac{1}{x} + \frac{1}{2x+1} - \frac{1}{x+1}\right)$

PARAMETRIC DIFFERENTIATION 1

1. In each case, a curve is defined by the given parametric equations.
Find an expression for $\frac{dy}{dx}$ in terms of the parameter t and simplify your answer where possible.

(a)	$x = 5t^2, \quad y = 10t$	(b)	$x = t^2 + 1, \quad y = 4t^3$
(c)	$x = 6t^2, \quad y = t^3$	(d)	$x = t^2 + t, \quad y = t^3$
(e)	$x = t^3 + t^2, \quad y = t^2 + t$	(f)	$x = (t+1)^2, \quad y = t^2 - 1$
(g)	$x = 4t^2 + 3, \quad y = 2t^3$	(h)	$x = t^2, \quad y = t(t^2 - 1)$
(i)	$x = t - \cos t, \quad y = 1 + \sin t$	(j)	$x = 2\sin t, \quad y = 2\cos t$
(k)	$x = at^2, \quad y = 2at$	(l)	$x = 3t^2 + 4t - 5, \quad y = t^3 + t^2$
(m)	$x = \sqrt{t}, \quad y = t^2 - 3$	(n)	$x = t^2, \quad y = \frac{1}{t}$
(o)	$x = e^t, \quad y = e^{2t}$	(p)	$x = 2t, \quad y = \frac{2}{t}$
(q)	$x = \frac{1}{t}, \quad y = \sqrt{t^2 + 1}$	(r)	$x = 4\cos t, \quad y = 2\sin t$
(s)	$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}$	(t)	$x = e^t \cos t, \quad y = e^t \sin t$
(u)	$x = \cos t, \quad y = \sin^2 t$	(v)	$x = \frac{2}{t+1}, \quad y = \frac{4}{t-1}$
(w)	$x = \frac{t-1}{t+1}, \quad y = \frac{2t-1}{t-2}$	(x)	$x = \frac{t}{t+1}, \quad y = \frac{t^2}{t+1}$
(y)	$x = \frac{1}{1+t}, \quad y = \frac{t}{1-t}$	(z)	$x = \frac{t}{1+t^2}, \quad y = \frac{t}{1-t^2}$

2. A curve is defined by the parametric equations

$$x = \sin\theta + \cos\theta, \quad y = \sin\theta - \cos\theta.$$

Show that $\frac{dy}{dx} = \frac{1 + \tan\theta}{1 - \tan\theta}$.

3. A curve is defined by the parametric equations

$$x = 4\sin\theta, \quad y = \cos 2\theta.$$

Show that $\frac{dy}{dx} = -\sin\theta$.
[You may assume without proof the identity $\sin 2\theta = 2\sin\theta\cos\theta$.]

4. A curve is defined by the parametric equations

$$x = \cos^2\theta, \quad y = \sin^3\theta.$$

Show that $\frac{dy}{dx} = -\frac{3}{2}\sin\theta$.

5. A curve is defined by the parametric equations

$$x = \sec\theta, \quad y = \tan\theta.$$

Show that $\frac{dy}{dx} = \operatorname{cosec}\theta$.

ANSWERS

1. (a) $\frac{1}{t}$ (b) $6t$ (c) $\frac{t}{4}$

(d) $\frac{3t^2}{2t+1}$ (e) $\frac{2t+1}{t(3t+2)}$ (f) $\frac{t}{t+1}$

(g) $\frac{3t}{4}$ (h) $\frac{3t^2-1}{2t}$ (i) $\frac{\cos t}{1+\sin t}$

(j) $-\tan t$ (k) $\frac{1}{t}$ (l) $\frac{t}{2}$

(m) $4t^{3/2}$ (n) $-\frac{1}{2t^3}$ (o) $2e^t$

(p) $-\frac{1}{t^2}$ (q) $-\frac{t^3}{\sqrt{t^2+1}}$ (r) $-\frac{1}{2}\cot t$

(s) $\frac{t^2+1}{t^2-1}$ (t) $\frac{\cos t+\sin t}{\cos t-\sin t}$ (u) $-2\cos t$

(v) $2\left(\frac{t+1}{t-1}\right)^2$ (w) $-\frac{3}{2}\left(\frac{t+1}{t-2}\right)^2$ (x) $t(t+2)$

(y) $-\left(\frac{1+t}{1-t}\right)^2$ (z) $\left(\frac{1+t^2}{1-t^2}\right)^3$

ADVANCED HIGHER MATHEMATICS

PARAMETRIC DIFFERENTIATION 2

1. A curve is defined by the parametric equations

$$x = t^2, \quad y = 2t.$$

Find the equation of the tangent to the curve at the point with parameter $t = 3$.

2. A curve is defined by the parametric equations

$$x = t^2 + 1, \quad y = 2t^3.$$

Find the equation of the tangent to the curve at the point with parameter $t = 1$.

3. A curve is defined by the parametric equations

$$x = t^2 + 2t - 1, \quad y = t^2 - 4t + 1.$$

Find the equation of the tangent to the curve at the point with parameter $t = 0$.

4. A curve is defined by the parametric equations

$$x = t^2 - 3, \quad y = 2t^3.$$

Find the equation of the tangent to the curve at the point with parameter $t = 1$.

5. A curve is defined by the parametric equations

$$x = t^2, \quad y = t(t^2 + 1).$$

Find the equation of the tangent to the curve at the point with parameter $t = 1$.

6. A curve is defined by the parametric equations

$$x = 2t^2 + t - 5, \quad y = t^2 + 3t + 1.$$

Find the equation of the tangent to the curve at the point with parameter $t = 1$.

7. A curve is defined by the parametric equations

$$x = t(t + 1), \quad y = t^2(t - 1).$$

Find the equation of the tangent to the curve at the point with parameter $t = 1$.

8. In each case, a curve is defined by the given parametric equations.
Find the coordinates of the stationary point(s) on each curve.
(You do not need to determine the nature of the stationary points.)

(a) $x = t^2 + t - 1, \quad y = t^2 - 4t + 1$ (b) $x = t^2 + 1, \quad y = 2t(t - 2)$

(c) $x = t^3 - 5t, \quad y = t^2 - 6t + 2$ (d) $x = t^2 + t, \quad y = t^3 - 12t$

(e) $x = t^2 + \frac{2}{t}, \quad y = t^2 - \frac{2}{t}$ (f) $x = t - \frac{1}{t}, \quad y = t + \frac{1}{t}$

(g) $x = (t^2 + 1)^2, \quad y = t(t - 2)$ (h) $x = t^2 + 1, \quad y = t(t - 3)^2$

9. In each case, a curve is defined by the given parametric equations.

Find an expression for the second derivative $\frac{d^2y}{dx^2}$ in terms of the parameter t.

(a) $x = t^2, \quad y = 2t$ (b) $x = t^2, \quad y = \ln t$

(c) $x = \frac{2}{t}, \quad y = 2t^3 + 1$ (d) $x = 2t^3 - 1, \quad y = 3t^2$

(e) $x = t^2, \quad y = \frac{4}{t}$ (f) $x = t^2 + t, \quad y = t^2 - t$

10. A curve is defined by the parametric equations

$$x = t + \sin t, \quad y = t - \cos t.$$

Show that $\frac{d^2y}{dx^2} = \frac{1 + \sin t + \cos t}{(1 + \cos t)^3}$.

11. A curve is defined by the parametric equations

$$x = t^2 + 1, \quad y = t(t^2 + 1).$$

Show that $\frac{d^2y}{dx^2} = \frac{3t^2 - 1}{4t^3}$.

ANSWERS

1. $3y = x + 9$ **2.** $y = 3x - 4$ **3.** $y = -2x - 1$

4. $y = 3x + 8$ **5.** $y = 2x$ **6.** $y = x + 7$

7. $3y = x - 2$

8. (a) (5, −3) (b) (2, −2)

(c) (12, −7) (d) (6, −16) and (2, 16)

(e) (−1, 3) (f) (0, 2) and (0, −2)

(g) (4, −1) (h) (2, 4) and (10, 0)

9. (a) $-\frac{1}{2t^3}$ (b) $-\frac{1}{2t^4}$ (c) $6t^5$

(d) $-\frac{1}{6t^3}$ (e) $\frac{3}{t^5}$ (f) $\frac{4}{(2t+1)^3}$

HOMEWORK ON FURTHER DIFFERENTIATION

1. Differentiate with respect to x and simplify where possible:

   (a) $y = \sin^{-1}(4x)$ (b) $y = \tan^{-1}(\sqrt{x^2 - 1})$

   (c) $y = (1 - x^2)\sin^{-1} x$ (d) $y = (4x^2 + 1)\tan^{-1}(2x)$

2. The equation $x^2 + y^2 = xy$ defines y implicitly in terms of x.
   Use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y.

3. The equation $y^3 + 3xy = 3x^2 - 5$ defines a curve passing through the point A(2, 1).

   (a) Use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y.
   (b) Find the equation of the tangent to the curve at A.

4. The equation $2x^2 + 4xy = 1$ defines y implicitly in terms of x.

   Use implicit differentiation to show that: (a) $\frac{dy}{dx} = \frac{-x - y}{x}$

   (b) $\frac{d^2y}{dx^2} = \frac{x + 2y}{x^2}$

5. Given $y = 10^x$, use logarithmic differentiation to find $\frac{dy}{dx}$ in terms of x.

6. Given $y = \frac{x^2}{\sqrt{x^4 + 1}}$, use logarithmic differentiation to find $\frac{dy}{dx}$ in terms of x.

7. A curve is defined by the parametric equations $x = (t + 1)^2$, $y = t^2 - 1$ for all t.
   Find an expression for $\frac{dy}{dx}$ in terms of t in its simplest form.

8. A curve is defined by the parametric equations $x = e^t \cos t$, $y = e^t \sin t$ for all t.
   Find an expression for $\frac{dy}{dx}$ in terms of t in its simplest form.

9. A curve is defined by the parametric equations $x = \frac{t^2}{t + 1}$, $y = \frac{t}{t + 1}$, $t \neq -1$.
   Find an expression for $\frac{dy}{dx}$ in terms of t in its simplest form.

10. A curve is defined by the parametric equations $x = t^2 + 1$, $y = t^4$ for all t.
    Find the equation of the tangent to the curve at the point with parameter $t = 1$.

11. A curve is defined by the parametric equations $x = (t^2 + 1)^2$, $y = t(t - 2)$ for all t.
    Find the cordinates of the stationary point on the curve.
    [You do not need to justify the nature of the stationary point.]

12. A curve is defined by the parametric equations $x = t(t + 1)$, $y = t(t - 1)$ for all t.
    Find the second derivative $\frac{d^2y}{dx^2}$ in terms of t in its simplest form.

13. A curve is defined by the parametric equations $x = t^2 + 1$, $y = t(t^2 + 1)$ for all t.
    Find the second derivative $\frac{d^2y}{dx^2}$ in terms of t in its simplest form.