DIFFERENTIAL EQUATIONS 2

1. Find the general solution of each differential equation, in case expressing y **explicitly** in terms of x.

(a) $\frac{dy}{dx} = y$	(b) $\frac{dy}{dx} = 2y$	(c) $\frac{dy}{dx} = \frac{y}{x}$
(d) $\frac{dy}{dx} = \frac{y}{x+1}$	(e) $\frac{dy}{dx} = \frac{y+1}{x+2}$	(f) $(x-3)\frac{dy}{dx} = y$
(g) $\frac{dy}{dx} = 6xy$	(h) $\frac{dy}{dx} = 3x^2(y+1)$	(i) $\frac{dy}{dx} = \frac{1+y}{1+x}$
(j) $\frac{dy}{dx} = y+1$	(k) $\frac{dy}{dx} = \frac{y+2}{x}$	(l) $\frac{dy}{dx} = \frac{y}{2x+1}$
(m) $\frac{dy}{dx} = \frac{y+1}{x}$	(n) $\frac{dy}{dx} = \frac{y+3}{x+2}$	(o) $\frac{dy}{dx} = \frac{y-1}{2x-1}$
(p) $\frac{dy}{dx} = x(y+1)$	(q) $\frac{dy}{dx} = -y$	(r) $\frac{dy}{dx} = \frac{y}{2x}$
(s) $\frac{dy}{dx} = \frac{y+1}{3x}$		

2. (a) Express $\frac{2x+1}{x(x+1)}$ in partial fractions.

   (b) Hence find the general solution of the differential equation

$$x(x+1)\frac{dy}{dx} = y(2x+1),$$

   expressing y explicitly in terms of x.

3. (a) Express $\frac{1}{x(x+1)}$ in partial fractions.

   (b) Hence find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{x(x+1)},$$

   expressing y explicitly in terms of x.

4. (a) Express $\frac{3x+4}{x(x+1)}$ in partial fractions.

   (b) Hence find the general solution of the differential equation

$$x(x+1)\frac{dy}{dx} = y(3x+4),$$

   expressing y explicitly in terms of x.

5. (a) Express $\frac{2x}{x^2-1}$ in partial fractions.

   (b) Hence find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{2xy}{x^2-1},$$

   expressing y explicitly in terms of x.

6. (a) Express $\frac{1}{x(x-1)}$ in partial fractions.

   (b) Hence find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{x(x-1)},$$

   expressing y explicitly in terms of x.

7. (a) Express $\frac{x}{(x+2)(x+1)}$ in partial fractions.

   (b) Hence find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{xy}{(x+2)(x+1)},$$

   expressing y explicitly in terms of x.

16. (a) Express $\dfrac{1}{(x+1)(x+3)}$ in partial fractions.

(b) Hence find the general solution of the differential equation

$$\frac{dy}{dx}=\frac{y}{(x+1)(x+3)},$$

expressing y in terms of x.

17. Study the worked example below carefully.

Worked Example

Express $\dfrac{2}{y(y+2)}$ in partial fractions and hence find the general solution of the differential equation

$$2\frac{dy}{dx}=y(y+2),$$

expressing y explicitly in terms of x.

Solution

Set $\dfrac{2}{y(y+2)}=\dfrac{A}{y}+\dfrac{B}{y+2}$

$=\dfrac{A(y+2)+By}{y(y+2)}$

$\Rightarrow \quad 2=A(y+2)+By$

Put $y=0 \Rightarrow \quad 2=A(2)$

$\Rightarrow \quad A=\dfrac{2}{2}=1$

Put $y=-2 \Rightarrow \quad 2=B(-2)$

$\Rightarrow \quad B=\dfrac{2}{-2}=-1$

So $\dfrac{2}{y(y+2)}=\dfrac{1}{y}-\dfrac{1}{y+2}$.

Now $2\dfrac{dy}{dx}=y(y+2)$

$\Rightarrow \quad 2dy=y(y+2)dx$

$\Rightarrow \quad \displaystyle\int\frac{2}{y(y+2)}dy=\int dx$

$\Rightarrow \quad \displaystyle\int\left(\frac{1}{y}-\frac{1}{y+2}\right)dy=\int dx$

$\Rightarrow \quad \ln y-\ln(y+2)=x+C$

$\Rightarrow \quad \ln\left(\dfrac{y}{y+2}\right)=x+C \qquad (e\quad)$

$\Rightarrow \quad \dfrac{y}{y+2}=e^{x}\cdot e^{C}$

$\Rightarrow \quad \dfrac{y}{y+2}=Ae^{x} \qquad (A=e^{C})$

$\Rightarrow \quad y=Ae^{x}(y+2)$

$\Rightarrow \quad y=Ae^{x}y+2Ae^{x}$

$\Rightarrow \quad y-Ae^{x}y=2Ae^{x}$

$\Rightarrow \quad y(1-Ae^{x})=2Ae^{x}$

$\Rightarrow \quad y=\dfrac{2Ae^{x}}{1-Ae^{x}}$

(a) Express $\dfrac{1}{y(y+1)}$ in partial fractions and hence find the general solution of the differential equation

$$\frac{dy}{dx}=2xy(y+1),$$

expressing y explicitly in terms of x.

(b) Express $\dfrac{2}{y^2-1}$ in partial fractions and hence find the general solution of the differential equation

$$2\frac{dy}{dx}=3x^2(y^2-1),$$

expressing y explicitly in terms of x.

ANSWERS

1. (a) $y = Ae^x$ (b) $y = Ae^{2x}$ (c) $y = Ax$
   (d) $y = A(x+1)$ (e) $y = A(x+2) - 1$ (f) $y = A(x-3)$
   (g) $y = Ae^{3x^2}$ (h) $y = Ae^{x^3} - 1$ (i) $y = A(1+x) - 1$
   (j) $y = Ae^x - 1$ (k) $y = Ax - 2$ (l) $y = A\sqrt{2x+1}$
   (m) $y = Ax - 1$ (n) $y = A(x+2) - 3$ (o) $y = A\sqrt{2x-1} + 1$
   (p) $y = Ae^{\frac{1}{2}x^2} - 1$ (q) $y = Ae^{-x}$ (r) $y = A\sqrt{x}$
   (s) $y = Ax^{\frac{1}{3}} - 1$

2. (a) $\frac{2x+1}{x(x+1)} = \frac{1}{x} + \frac{1}{x+1}$ (b) $y = Ax(x+1)$

3. (a) $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$ (b) $y = \frac{Ax}{x+1}$

4. (a) $\frac{3x+4}{x(x+1)} = \frac{3}{x} + \frac{1}{x+1}$ (b) $y = Ax^3(x+1)$

5. (a) $\frac{2x}{x^2-1} = \frac{1}{x-1} + \frac{1}{x+1}$ (b) $y = A(x-1)(x+1)$ $\left[= A(x^2-1)\right]$

6. (a) $\frac{1}{x(x-1)} = -\frac{1}{x} + \frac{1}{x-1}$ (b) $y = \frac{A(x-1)}{x}$

7. (a) $\frac{x}{(x+2)(x+1)} = \frac{2}{x+2} - \frac{1}{x+1}$ (b) $y = \frac{A(x+2)^2}{x+1}$

8. (a) $\frac{3-2x}{x(1-x)} = \frac{3}{x} + \frac{1}{1-x}$ (b) $y = \frac{Ax^3}{1-x}$

9. (a) $\frac{2}{(x+1)(x+3)} = \frac{1}{x+1} - \frac{1}{x+3}$ (b) $y = \frac{A(x+1)}{x+3}$

10. (a) $\frac{2(x+1)}{x(x+2)} = \frac{1}{x} + \frac{1}{x+2}$ (b) $y = Ax(x+2)$

11. (a) $\frac{x+3}{(x+1)(x+2)} = \frac{2}{x+1} - \frac{1}{x+2}$ (b) $y = \frac{A(x+1)^2}{x+2}$

12. (a) $\frac{x+2}{x(x+1)} = \frac{2}{x} - \frac{1}{x+1}$ (b) $y = \frac{Ax^2}{x+1}$

13. (a) $\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}$ (b) $y = \frac{A(x-1)}{x+1}$

14. (a) $\frac{3x-1}{x(2x-1)} = \frac{1}{x} + \frac{1}{2x-1}$ (b) $y = Ax\sqrt{2x-1}$

15. (a) $\frac{x+1}{x(x+2)} = \frac{1/2}{x} + \frac{1/2}{x+2}$ $\left[= \frac{1}{2x} + \frac{1}{2(x+2)}\right]$
    (b) $y = A\sqrt{x(x+2)}$ $\left[= A\sqrt{x^2+2x}\right]$

16. (a) $\frac{1}{(x+1)(x+3)} = \frac{1/2}{x+1} - \frac{1/2}{x+3}$ $\left[= \frac{1}{2(x+1)} - \frac{1}{2(x+3)}\right]$ (b) $y = A\sqrt{\frac{x+1}{x+3}}$

17. (a) $\frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}$; $y = \frac{Ae^{x^2}}{1-Ae^{x^2}}$
    (b) $\frac{2}{y^2-1} = \frac{1}{y-1} - \frac{1}{y+1}$; $y = \frac{1+Ae^{x^3}}{1-Ae^{x^3}}$

18. (a) $y = Axe^x$ (b) $y = \frac{Ae^x}{x+1}$

PARTICULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS

1. Find the particular solution of each differential equation, in each case expressing y **explicitly** in terms of x.

(a) $3y^2\frac{dy}{dx} = 2x+1$; $y = 2$ when $x = 2$

(b) $y^2\frac{dy}{dx} = 2x^2+1$; $y = 1$ when $x = 1$

(c) $y^4\frac{dy}{dx} = 3x^2$; $y = -1$ when $x = 1$

(d) $e^y\frac{dy}{dx} + \sin x = 0$; $y = 0$ when $x = \frac{\pi}{2}$

(e) $\frac{dy}{dx} = y$; $y = 2$ when $x = 0$

(f) $\frac{dy}{dx} = 4xy$; $y = 4$ when $x = 0$

(g) $x\frac{dy}{dx} = y+2$; $y = 7$ when $x = 3$

(h) $(x+1)\frac{dy}{dx} = y$; $y = 12$ when $x = 2$

(i) $(x+2)\frac{dy}{dx} = y+1$; $y = -2$ when $x = -3$

(j) $\frac{dy}{dx} = 6x(y+1)$; $y = 3$ when $x = 0$

(k) $(x+4)\frac{dy}{dx} + 3 = y$; $y = 13$ when $x = 1$

(l) $\frac{dy}{dx} - 2xy = 0$; $y = 3$ when $x = 0$

(m) $\frac{dy}{dx} = \frac{y+1}{x+4}$; $y = 3$ when $x = -2$

(n) $\frac{dy}{dx} = y^2$; $y = -1$ when $x = 3$

(o) $(x+1)\frac{dy}{dx} + 2 = y$; $y = 8$ when $x = 1$

(p) $\frac{dy}{dx} = \frac{y+2}{x+1}$; $y = -2$ when $x = -1$

(q) $x\frac{dy}{dx} + y^2 = 0 \quad (x > 0)$; $y = \frac{1}{2}$ when $x = 1$

(r) $(x+2)\frac{dy}{dx} + 5 = y$; $y = 1$ when $x = 2$

2. (a) Express $\frac{1}{y(y+1)}$ in partial fractions.

(b) Hence solve the differential equation

$$\frac{dy}{dx} = \frac{y(y+1)}{x},$$

given that $y = 4$ when $x = 2$, expressing y explicitly in terms of x.

3. (a) Express $\frac{2}{y^2-1}$ in partial fractions.

(b) Hence solve the differential equation

$$2x\frac{dy}{dx} + 1 = y^2,$$

given that $y = -3$ when $x = 1$, expressing y explicitly in terms of x.

4. (a) Express $\frac{1}{(y+1)(y+2)}$ in partial fractions.

(b) Hence show that the general solution of the differential equation

$$\frac{dy}{dx} = \frac{(y+1)(y+2)}{x}$$

can be expressed in the form $y = \frac{2Ax-1}{1-Ax}$, where A is a constant.

Find the particular solution of this differential equation for which $y = -3$ when $x = 1$.

5. (a) Express $\frac{4}{y^2-4}$ in partial fractions.

(b) Hence show that the general solution of the differential equation

$$4\left(\frac{dy}{dx}+1\right)=y^2$$

can be expressed in the form $y=\frac{2(1+Ae^x)}{1-Ae^x}$, where A is a constant.

Find the particular solution of this differential equation for which $y=-4$ when $x=0$.

6. (a) Express $\frac{2}{y(y+2)}$ in partial fractions.

(b) Hence solve the differential equation

$$2x\frac{dy}{dx}=y(y+2),$$

given that $y=2$ when $x=1$, expressing y explicitly in terms of x.

7. (a) Express $\frac{1}{y(1-y)}$ in partial fractions.

(b) Hence solve the differential equation

$$\frac{dy}{dx}=\frac{2x}{y(1-y)},$$

given that $y=\frac{2}{3}$ when $x=0$.

8. At any point (x, y) on a curve with equation $y=f(x)$, the gradient of the tangent to the curve is given by

$$\frac{dy}{dx}=\frac{x^2+1}{y^2}, \quad y\neq 0.$$

The point $P(-1, 1)$ is on the curve.

(a) Find the equation of the tangent to the curve at P.
(b) Solve the differential equation to find the equation of the curve in the form $y=f(x)$.

9. (a) Express $\frac{1}{x(x+1)}$ in partial fractions.

(b) At any point (x, y) on a curve with equation $y=f(x)$ where $x>0$, the gradient of the tangent to the curve is given by

$$\frac{dy}{dx}=\frac{y}{x(x+1)}.$$

The point $P(3, 6)$ is on the curve.

(i) Find the equation of the tangent to the curve at P.
(ii) Solve the differential equation to find the equation of the curve in the form $y=f(x)$.

10. (a) Express $\frac{3}{x^2-x-2}$ in partial fractions.

(b) Hence find the general solution of the differential equation

$$\frac{dy}{dx}=\frac{3y}{x^2-x-2}$$

in the region where $x>2$, expressing y explicitly in terms of x.

Find also the particular solution of this differential equation for which $y=1$ when $x=5$.

ANSWERS

1. (a) $y=(x^2+x+2)^{\frac{1}{3}}$ (b) $y=(2x^3+3x-4)^{\frac{1}{3}}$ (c) $y=(5x^3-6)^{\frac{1}{5}}$
   (d) $y=\ln(\cos x+1)$ (e) $y=2e^x$ (f) $y=4e^{2x^2}$
   (g) $y=3x-2$ (h) $y=4(x+1)$ (i) $y=x+1$
   (j) $y=4e^{3x^2}-1$ (k) $y=2x+11$ (l) $y=3e^{x^2}$
   (m) $y=2x+7$ (n) $y=\dfrac{1}{2-x}$ (o) $y=3x+5$
   (p) $y=5x+3$ (q) $y=\dfrac{1}{\ln x+2}$ (r) $y=3-x$

2. (a) $\dfrac{1}{y(y+1)}=\dfrac{1}{y}-\dfrac{1}{y+1}$ (b) $y=\dfrac{2x}{5-2x}$

3. (a) $\dfrac{2}{y^2-1}=\dfrac{1}{y-1}-\dfrac{1}{y+1}$ (b) $y=\dfrac{1+2x}{1-2x}$

4. (a) $\dfrac{1}{(y+1)(y+2)}=\dfrac{1}{y+1}-\dfrac{1}{y+2}$ (b) $y=\dfrac{4x-1}{1-2x}$

5. (a) $\dfrac{4}{y^2-4}=\dfrac{1}{y-2}-\dfrac{1}{y+2}$ (b) $y=\dfrac{2(1+3e^x)}{1-3e^x}$

6. (a) $\dfrac{2}{y(y+2)}=\dfrac{1}{y}-\dfrac{1}{y+2}$ (b) $y=\dfrac{2x}{2-x}$

7. (a) $\dfrac{1}{y(1-y)}=\dfrac{1}{y}+\dfrac{1}{1-y}$ (b) $y=\dfrac{2e^{x^2}}{1+2e^{x^2}}$

8. (a) $y=2x+3$ (b) $y=(x^3+3x+5)^{\frac{1}{3}}$

9. (a) $\dfrac{1}{x(x+1)}=\dfrac{1}{x}-\dfrac{1}{x+1}$ (b)(i) $2y=x+9$ (ii) $y=\dfrac{8x}{x+1}$

10. (a) $\dfrac{3}{x^2-x-2}=\dfrac{1}{x-2}-\dfrac{1}{x+1}$ (b) $y=\dfrac{A(x-2)}{x+1}$; $y=\dfrac{2(x-2)}{x+1}$

DIFFERENTIAL EQUATIONS AS MATHEMATICAL MODELS

1. In a certain culture of bacteria, the rate of increase of bacteria at any given time is proportional to the number of bacteria present at that time. If x denotes the number of bacteria present after t hours of observation, the growth of bacteria in the culture is governed by a differential equation of the form

$$\frac{dx}{dt} = kx,$$

where k is a constant.

(a) Find the general solution of this differential equation, expressing x as a function of t.

(b) Given that the number of bacteria increased from 600 at time $t = 0$ to 1800 at time $t = 2$, find the number of bacteria present at the end of 4 hours (correct to the nearest whole number).

2. A scientist grows a culture of bacteria in his laboratory. The number of bacteria, N, after t hours is assumed to satisfy the differential equation

$$\frac{dN}{dt} = kN,$$

where k is a constant.

There are initially 80 bacteria in the culture, and the scientist observes that there are 197 bacteria after 3 hours.

How long will it take for the bacteria population to increase to 10 times the number initially present?
(Give your answer correct to the nearest $0 \cdot 1$ hour.)

3. A radioactive isotope of plutonium decays exponentially such that the rate of decrease of the mass of the isotope at any given time is proportional to the mass remaining at that time. If m denotes the mass of the isotope (in grams) remaining t years after decay began, the decay is governed by a differential equation of the form

$$\frac{dm}{dt} = -km,$$

where k is a positive constant.

(a) Find the general solution of this differential equation, expressing m as a function of t.

(b) Given that this isotope has a half-life of approximately 139 years (that is, given any initial mass of the isotope, one-half will disintegrate in 139 years), show that starting with 120 grams of this isotope, the mass remaining after t years is given approximately by

$$m = 120e^{-0\cdot005t}.$$

Hence find the mass remaining after 50 years.

4. A rumour is spread among a large population of students. It is assumed that the rate at which the rumour is spreading is proportional to the number of students who have heard the rumour. Hence, if N is the number of students who have heard the rumour after t days, N satisfies a differential equation of the form

$$\frac{dN}{dt} = kN,$$

where k is a constant.

The rumour is spread initially by 2 students and, after five days, 985 students had heard the rumour.

(a) Solve this differential equation and show that, approximately,

$$N = 2e^{1\cdot24t}.$$

(b) After how many days had 250 students heard the rumour?
(Give your answer to the nearest $0 \cdot 1$ day.)

5. A television producer predicts that viewing figures for her programme will increase at a rate proportional to the number of viewers. Hence, if n is the number of viewers after t weeks, it can be assumed that n satisfies a differential equation of the form

$$\frac{dn}{dt} = kn,$$

where k is a constant.

(a) Find the general solution of this differential equation, expressing n as a function of t.

The number of viewers at the moment (time $t = 0$) is 100 000 and the producer predicts that there will be 120 925 viewers in two weeks' time.

(b) (i) Find the value of the constant k correct to three decimal places.

(ii) The producer is told that the target audience for her programme is 500 000 viewers.
How long should it take to reach this target, to the nearest $0 \cdot 1$ week?

6. A metal plate that has been heated cools from $180°C$ to $150°C$ in 20 minutes when surrounded by air at a temperature of $60°C$. Newton's Law of Cooling states that if T is the temperature of the plate (in $°C$) after t minutes of cooling, then T satisfies a differential equation of the form

$$\frac{dT}{dt} = -k(T - 60),$$

where k is a constant.

(a) Find the temperature of the plate at the end of one hour of cooling, correct to the nearest $0 \cdot 1°C$.

(b) When will the temperature of the plate reach $100°C$, correct to the nearest minute?

7. If the temperature is constant, then the rate of change of barometric pressure p with respect to altitude h is proportional to p, where p and h are measured in appropriate units. The barometric pressure is thus governed by a differential equation of the form

$$\frac{dp}{dh} = kp,$$

where k is a constant.

If $p = 30$ at sea-level ($h = 0$) and $p = 29$ when $h = 1$, find p when $h = 5$, correct to the nearest $0 \cdot 1$ unit.

8. In a certain culture of bacteria, the rate of increase of bacteria at any given time is proportional to the number of bacteria present at that time. Thus, if x denotes the number of bacteria present after t hours of observation, then x satisfies a differential equation of the form

$$\frac{dx}{dt} = kx,$$

where k is a constant.

Given that the number of bacteria increased from 5000 (at time $t = 0$) to 15 000 in 10 hours, estimate the number of bacteria at the end of 20 hours, giving your answer correct to the nearest thousand.

How many hours will it take for the number of bacteria to grow to 50 000? (Give your answer correct to the nearest $0 \cdot 1$ hour.)

9. The polonium isotope ^{210}Po decays such that the rate of decrease of the mass of this isotope at any given time is proportional to the mass remaining. If m denotes the mass of this isotope (in mg) remaining after t days of decay, the decay is modelled by a differential equation of the form

$$\frac{dm}{dt} = -km,$$

where k is a positive constant.

Given that the isotope has a half-life (see question 3) of approximately 140 days, estimate the mass which will remain from a 20 mg sample after 2 weeks, giving your answer correct to the nearest $0 \cdot 1$ mg.

10. The rate at which a body loses speed at any given time t seconds as it travels through a resistive medium is given by kv metres per second per second, where v metres per second is the speed of the body at that instant, and k is a positive constant. The speed of this body is governed by the differential equation

$$\frac{dv}{dt} = -kv.$$

(a) Find the general solution of this differential equation, expressing v as a function of t.

(b) Show that if the initial speed of the body is u metres per second, then the time taken for the body to decrease its speed to $\frac{1}{2}u$ metres per second is $\frac{1}{k}\ln 2$ seconds.

11. The rate at which a population of mice is increasing at any given time is assumed to be proportional to the number of mice.

(a) If m denotes the number of mice after t months, write down a differential equation satisfied by m and find the general solution, expressing m as a function of t.

(b) Given that at time $t = 0$ there are 6 mice and it takes 6 months for the number of mice to double, after how many months will there be 90 mice (correct to the nearest $0 \cdot 1$ month)?

12. When taken from the oven, the temperature, $T°C$, of a bowl of pasta after t minutes of cooling is such that

$$\frac{dT}{dt} = -kT,$$

where k is a positive constant.

The bowl of pasta was at a temperature of $70°C$ after one minute of cooling, and was at a temperature $49°C$ after a further minute of cooling.

(a) Find the temperature of the pasta on leaving the oven, correct to the nearest $°C$.

(b) How long, to the nearest $0 \cdot 1$ minute, will the pasta take to cool down to room temperature, given that room temperature is $20°C$?

13. When a valve is opened, the rate at which the water drains from a pool is proportional to the square root of the depth of the water. This can be represented by the differential equation

$$\frac{dh}{dt} = -k\sqrt{h},$$

where h is the depth (in metres) of the water and t is the time (in minutes) elapsed since the valve was opened, and k is a positive constant.

(a) Find the general solution of the differential equation.
(You need **not** express h explicitly as a function of t.)

(b) Given that the pool was initially 4 metres deep, and that the depth of water was 1 metre after 20 minutes of draining, how long did it take for the pool to drain completely?

14. In a town with a large population, a flu virus is spreading rapidly. The percentage P infected t days after the initial outbreak satisfies the differential equation

$$\frac{dP}{dt} = kP,$$

where k is a constant.

$0 \cdot 5\%$ of the population are infected initially, and $2 \cdot 5\%$ are infected after 7 days.

Find the value of k correct to 4 decimal places and verify that about $12 \cdot 5\%$ of the population will be infected after 14 days.

15. (a) Express $\dfrac{1}{x(1-x)}$ in partial fractions.

(b) The spread of a disease in a large population can be modelled by means of a differential equation. The proportion x of the population infected with the disease after t days satisfies a differential equation of the form

$$\frac{dx}{dt} = kx(1-x),$$

where k is a constant.

Find the general solution of the differential equation, expressing x explicitly as a function of t.

Given that $x = \dfrac{1}{1000}$ when $t = 0$ and $x = \dfrac{1}{100}$ when $t = 5$, verify that about 9% of the population was infected after ten days, and find after how many days 25% of the population will become infected.

16. In a chemical reaction, two substances X and Y combine to form a third substance Z. Let Q denote the number of grams of Z formed t minutes after the reaction begins. The rate at which Q varies is represented by the differential equation

$$\frac{dQ}{dt} = \frac{(30-Q)(15-Q)}{900}.$$

(a) Express $\dfrac{900}{(30-Q)(15-Q)}$ in partial fractions.

(b) Use your answer to (a) to find the general solution of the differential equation, expressing Q explicitly as a function of t.

Given that $Q = 0$ when $t = 0$, find, correct to two decimal places:
(i) the number of grams of Z formed 45 minutes after the reaction begins
(ii) the time taken to form 5 grams of Z.

17. (a) Express $\dfrac{1}{p(1-p)}$ in partial fractions.

(b) The spread of a disease in a large population can be modelled by means of a differential equation. The proportion p of the population infected with the disease after t days satisfies a differential equation of the form

$$\frac{dp}{dt} = kp(1-p).$$

$p = \dfrac{1}{500}$ when $t = 0$, and $p = \dfrac{1}{100}$ when $t = 2$.

(i) Express p explicitly as a function of t.
(ii) Verify that about 10% of the population was infected after five days.
(iii) How long will it take for 50% of the population to become infected?

ANSWERS

1. (a) $x = Ae^{kt}$ (b) 5400

2. 7·7 hours

3. (a) $m = Ae^{-kt}$ (b) 93·5 grams

4. (b) 3·9 days

5. (a) $n = Ae^{kt}$ (b)(i) $k = 0{\cdot}095$ (ii) 16·9 weeks

6. (a) $110{\cdot}6°C$ (b) 76 mins (or 1 hour 6 mins)

7. $p = 25{\cdot}3$

8. 45 000; 21·0 hours

9. 18·6 mg

10. (a) $v = Ae^{-kt}$

11. (a) $\dfrac{dm}{dt} = km$, where k is a constant (b) 23·4 months

12. (a) $100°C$ (b) 4·5 mins

13. (a) $2\sqrt{h} = C - kt$ (b) 40 mins

14. $k = 0{\cdot}2299$

15. (a) $\dfrac{1}{x(1-x)} = \dfrac{1}{x} + \dfrac{1}{1-x}$ (b) $x = \dfrac{Ae^{kt}}{1+Ae^{kt}}$; 12·6 days

16. (a) $\dfrac{900}{(30-Q)(15-Q)} = -\dfrac{60}{30-Q} + \dfrac{60}{15-Q}$

(b) $Q = \dfrac{15\left(Ae^{\frac{t}{60}} - 2\right)}{Ae^{\frac{t}{60}} - 1}$; (i) 10·36 grams (ii) 13·39 mins

17. (a) $\dfrac{1}{p(1-p)} = \dfrac{1}{p} + \dfrac{1}{1-p}$

(b)(i) $p = \dfrac{e^{0{\cdot}8087t}}{499 + e^{0{\cdot}8087t}}$ (iii) 7·7 days

DIFFERENTIAL EQUATIONS: HOMEWORK 1

1. Find the particular solution of each differential equation below, in each case expressing y explicitly in terms of x.

(a) $\dfrac{dy}{dx} = \dfrac{x(x-2)}{y^2}$; $y = 1$ when $x = 2$

(b) $e^{2y}\dfrac{dy}{dx} = x + 1$; $y = 0$ when $x = 1$

(c) $\sqrt{x}\dfrac{dy}{dx} - y = 0$; $y = 3$ when $x = 0$

(d) $\dfrac{dy}{dx} = \dfrac{y-1}{x+3}$; $y = 3$ when $x = -2$

(e) $(2x+1)\dfrac{dy}{dx} + y = 0$; $y = 2$ when $x = 4$

2. (a) Use integration by parts to find $\int x\cos x\,dx$.

(b) Hence find the general solution of the differential equation

$$e^{y}\frac{dy}{dx} - x\cos x = 0.$$

(c) Find the particular solution of the differential equation in (b) if $y = 0$ when $x = \pi$, expressing y explicitly in terms of x.

3. (a) Express $\dfrac{2}{x(x+2)}$ in partial fractions.

(b) Hence solve the differential equation

$$\frac{dy}{dx} = \frac{2y}{x(x+2)},$$

given that $y = 2$ when $x = 4$, expressing y explicitly in terms of x.

4. (a) Express $\dfrac{3(x-2)}{x(x-3)}$ in partial fractions.

(b) Hence solve the differential equation

$$x(x-3)\frac{dy}{dx} = 3y(x-2),$$

given that $y = 8$ when $x = 4$, expressing y explicitly in terms of x.

5. (a) Express $\dfrac{1}{y(y+1)}$ in partial fractions.

(b) Hence solve the differential equation

$$\frac{dy}{dx} = 2y(y+1),$$

given that $y = 3$ when $x = 0$, expressing y explicitly in terms of x.

6. (a) Express $\dfrac{1}{(y+1)(y+2)}$ in partial fractions.

(b) Hence show that the solution of the differential equation

$$\frac{dy}{dx} = \frac{3(y+1)(y+2)}{x}$$

can be expressed in the form $y = \dfrac{2Ax^3 - 1}{1 - Ax^3}$, where A is a constant.

(c) Find the particular solution of the differential equation in (b) if $y = -3$ when $x = 1$.

DIFFERENTIAL EQUATIONS: HOMEWORK 2

1. The number of strands of bacteria, N, present in a culture after t days of growth is assumed to be increasing at a rate proportional to the number of strands present.

(a) Write down a differential equation satisfied by N and hence express N as a function of t.

(b) Given that there are 200 strands initially present, and that the number of strands observed after one week is 3850, estimate the number of strands likely to be present after two weeks, to the nearest thousand.

2. An outdoor thermometer registering a temperature of $40°$ is brought into a room where the temperature is $70°$. The temperature registered on the thermometer will gradually increase from $40°$ until the thermometer eventually registers the room temperature of $70°$.

Let $T°$ be the temperature registered on the thermometer t minutes after it is brought into the room. The rate at which T varies is represented by the differential equation

$$\frac{dT}{dt} = k(70 - T),$$

where k is a constant.

Given that the thermometer registers $60°$ after 5 minutes, after how many minutes will the thermometer register a temperature of $65°$?

3. A scientist grows a culture of bacteria in his laboratory. The number of bacteria, B, after t hours is assumed to satisfy the differential equation

$$\frac{dB}{dt} = kB,$$

where k is a constant.

Records indicate that there are 650 bacteria after 2 hours, and 900 bacteria after 5 hours.

(a) Find the initial number of bacteria in the culture.

(b) Estimate the number of bacteria at the end of the first day of growth.

4. When a valve is opened, the rate at which the water drains from a pool is proportional to the square root of the depth of the water. Let h be the depth (in metres) of the water t minutes after the valve was opened.

(a) Write down a differential equation satisfied by h and find the general solution of your differential equation.
(You need **not** express h explicitly as a function of t.)

(b) Given that the pool was initially 9 metres deep, and that the depth of water was $2 \cdot 25$ metres after half an hour of draining:
(i) find the depth of water after 12 minutes of draining
(ii) how long will it take for for the pool to drain to a depth of 1 metre?

5. (a) Express $\dfrac{1}{x(1-x)}$ in partial fractions.

(b) The spread of a disease in a large population can be modelled by means of a differential equation. The **proportion** x of the population infected with the disease after t days satisfies a differential equation of the form

$$\frac{dx}{dt} = kx(1-x), \qquad \text{where } k \text{ is a constant.}$$

(i) Find the general solution of the differential equation, expressing x explicitly as a function of t.

(ii) Given that $x = \frac{1}{250}$ when $t = 0$ and $x = \frac{1}{100}$ when $t = 3$, verify that about 8% of the population was infected after ten days, and find after how many days 20% of the population will become infected.

6. In a chemical reaction, two substances X and Y combine to form a third substance Z. Let Q denote the number of grams of Z formed t minutes after the reaction begins. The rate at which Q varies is represented by the differential equation

$$\frac{dQ}{dt} = \frac{(40-Q)(20-Q)}{600}.$$

(a) Express $\dfrac{600}{(40-Q)(20-Q)}$ in partial fractions.

(b) Use your answer to (a) to find the general solution of the differential equation, expressing Q explicitly as a function of t.

Given that $Q = 0$ when $t = 0$, find, correct to one decimal place:
(i) the number of grams of Z formed one hour after the reaction begins
(ii) the time taken to form 10 grams of Z.

EXAMINATION QUESTIONS ON DIFFERENTIAL EQUATIONS WITH VARIABLES SEPARABLE

1. A mathematical biologist believes that the differential equation

$$y\frac{dy}{dx} - 3x = x^4$$

models a process.

Given that $y = 2$ when $x = 1$, obtain y in terms of x.

2. (a) A function $y(x)$ satisfies the differential equation $\frac{dy}{dx} = \frac{y}{x}$.

Given that $y = 2$ when $x = 1$, obtain y as a function of x.

(b) A function $x(t)$ satisfies the differential equation $\frac{dx}{dt} = -2x^3$.

Given that $x = 1$ when $t = 0$, obtain x as a function of t for $t \geq 0$

3. The number of strands of bacteria, $B(t)$, present in a culture after t days of growth is assumed to be increasing at a rate proportional to the number of bacteria present.

(a) Write down a differential equation for B and find the general solution for B in terms of t.

(b) Given that the number of strands observed after 1 day is 502 and after 4 days is 1833, find the number of strands initially present.

4. A chemical plant food loses effectiveness at a rate proportional to the amount present in the soil. The amount M grams of plant food effective after t days satisfies the differential equation

$$\frac{dM}{dt} = kM, \quad \text{where } k \text{ is a constant.}$$

(a) Find the general solution for M in terms of t where the initial amount of plant food is 100 grams.

(b) Find the value of k if after 30 days only half the initial amount of plant food is effective.

(c) What percentage of the original amount of plant food is effective after 35 days?

(d) The plant food has to be renewed when its effectiveness falls below 25%. Is the manufacturer of the plant food justified in calling its product "sixty day super food"?

5. When a valve is opened, the rate at which the water drains from a pool is proportional to the square root of the depth of the water.

This can be represented by the differential equation

$$\frac{dh}{dt} = -\frac{\sqrt{h}}{10}, \quad h \geq 0,$$

where h is the depth (in metres) of the water and t is the time (in minutes) elapsed since the valve was opened.

(a) Express h as a function of t given that the pool was initially 9 metres deep.

(b) How long did it take to drain the pool?

6. The volume $V(t)$ of a cell at time t changes according to the law

$$\frac{dV}{dt} = V(10 - V) \qquad \text{for } 0 < V < 10.$$

(a) Show that

$$\frac{1}{10}\ln V - \frac{1}{10}\ln(10 - V) = t + C$$

for some constant C.

(b) Given that $V(0) = 5$, show that

$$V(t) = \frac{10e^{10t}}{1 + e^{10t}}.$$

(c) Obtain the limiting value of $V(t)$ as $t \to \infty$