FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS

1. Find the general solution of each differential equation, in each case expressing y **explicitly** in terms of x.

(a) $\frac{dy}{dx}+2y=1$ (b) $\frac{dy}{dx}+y=2e^x$

(c) $\frac{dy}{dx}+2y=e^{2x}$ (d) $\frac{dy}{dx}-y=-2e^{-x}$

(e) $\frac{dy}{dx}+2y=e^{-x}$ (f) $\frac{dy}{dx}-2y=6e^{-x}$

(g) $\frac{dy}{dx}+\frac{y}{x}=4x^2$ (h) $\frac{dy}{dx}+\frac{2y}{x}=x$

(i) $\frac{dy}{dx}-\frac{2y}{x}=2$ (j) $\frac{dy}{dx}-2y=e^{3x}$

(k) $\frac{dy}{dx}+2xy=e^{-x^2}\cos x$ (l) $\frac{dy}{dx}+\frac{y}{x}=x$

2. Rearrange each differential equation into the standard form $\frac{dy}{dx}+P(x)\cdot y=Q(x)$ and hence find the general solution of each equation, expressing y **explicitly** in terms of x.

(a) $e^x\frac{dy}{dx}+e^x y=\cos 2x$ (b) $x^2\frac{dy}{dx}-xy=4$

(c) $x\frac{dy}{dx}+y=x^2$ (d) $x\frac{dy}{dx}+y=\sin x$

(e) $x\frac{dy}{dx}+2y=\frac{4}{x}$ (f) $x\frac{dy}{dx}-2y=6x^4$

(g) $2\frac{dy}{dx}+y=e^x$ (h) $3x\frac{dy}{dx}+3y=2x$

(i) $x\frac{dy}{dx}-2y=x$ (j) $x^2\frac{dy}{dx}+2xy=\cos x$

(k) $x\frac{dy}{dx}-2y=-3x$

3. (a) Use the method of integration by parts to find $\int x\sin x\,dx$.

(b) Find the general solution of the differential equation

$$x\frac{dy}{dx}+y=x\sin x,$$

expressing y explicitly in terms of x.

4. (a) By writing $\tan x$ as $\frac{\sin x}{\cos x}$ and using a suitable substitution, show that

$$\int\tan x\,dx=\ln(\sec x)+C.$$

(b) Find the general solution of the differential equation

$$\cos x\frac{dy}{dx}+y\sin x=\cos^2 x,$$

expressing y explicitly in terms of x.

5. (a) By writing $\cot x$ as $\frac{\cos x}{\sin x}$ and using a suitable substitution, show that

$$\int\cot x\,dx=\ln(\sin x)+C.$$

(b) Find the general solution of the differential equation

$$\frac{dy}{dx}+y\cot x=\cot x,$$

expressing y explicitly in terms of x.

6. Find the particular solution of each differential equation subject to the given condition in brackets.

(a) $\frac{dy}{dx}+y=1$ ($y=3$ when $x=0$)

(b) $x\frac{dy}{dx}+3y=4x$ ($y=4$ when $x=1$)

(c) $x\frac{dy}{dx}+3y=5x^2$ ($y=0$ when $x=1$)

(d) $x^2\frac{dy}{dx}+2xy=1$ ($y=2$ when $x=1$)

(e) $x\frac{dy}{dx}-y=x^2$ ($y=0$ when $x=1$)

(f) $x\frac{dy}{dx}-y=x^2\cos x$ ($y=0$ when $x=\frac{\pi}{2}$)

(g) $\frac{dy}{dx}-y\cos x=2xe^{\sin x}$ ($y=1$ when $x=0$)

(h) $x\frac{dy}{dx}+2y=6x$ ($y=1$ when $x=1$)

(i) $\frac{dy}{dx}+3x^2y=6x^2$ ($y=1$ when $x=0$)

(j) $x\frac{dy}{dx} + 2y = 6x^4$ ($y = 5$ when $x = 1$)

7. (a) Use the method of integration by parts to find $\int xe^x dx$.

(b) Find the particular solution of the differential equation

$$\frac{dy}{dx} + y = x,$$

given that $y = 1$ when $x = 0$.

8. (a) By writing $\tan x$ as $\frac{\sin x}{\cos x}$ and using a suitable substitution, show that

$$\int \tan x dx = \ln(\sec x) + C.$$

(b)(i) Show that the general solution of the differential equation

$$\frac{dy}{dx} + y\tan x = \sec x$$

can be written in the form $y = \sin x + C\cos x$, where C is a constant.

(ii) Find the particular solution of this differential equation given that $y = 1$ when $x = 0$.

9. (a) Use the method of integration by parts to find $\int \ln x dx$.

(b) Find the particular solution of the differential equation

$$x\frac{dy}{dx} + y = \ln x,$$

given that $y = 1$ when $x = 1$.

10. (a) By writing $\tan x$ as $\frac{\sin x}{\cos x}$ and using a suitable substitution, show that

$$\int \tan x dx = \ln(\sec x) + C.$$

(b)(i) Show that the general solution of the differential equation

$$\frac{dy}{dx} + y\tan x = \cos^2 x$$

can be written in the form $y = (\sin x + C)\cos x$, where C is a constant.

(ii) Find the particular solution of this differential equation given that $y = \frac{3}{2}$ when $x = \frac{\pi}{4}$.

ANSWERS

(1) (a) $y = \frac{1}{2} + \frac{C}{e^{2x}}$ (b) $y = e^{x} + \frac{C}{e^{x}}$

(c) $y = \frac{1}{4}e^{2x} + \frac{C}{e^{2x}}$ (d) $y = e^{-x} + Ce^{x}$

(e) $y = \frac{1}{e^{x}} + \frac{C}{e^{2x}}$ (f) $y = -2e^{-x} + Ce^{2x}$

(g) $y = x^3 + \frac{C}{x}$ (h) $y = \frac{1}{4}x^2 + \frac{C}{x^2}$

(i) $y = -2x + Cx^2$ (j) $y = e^{3x} + Ce^{2x}$

(k) $y = \frac{\sin x + C}{e^{x^2}}$ (l) $y = \frac{1}{3}x^2 + \frac{C}{x}$

(2) (a) $y = \frac{\sin 2x}{2e^{x}} + \frac{C}{e^{x}}$ (b) $y = -\frac{2}{x} + Cx$

(c) $y = \frac{1}{3}x^2 + \frac{C}{x}$ (d) $y = \frac{C - \cos x}{x}$

(e) $y = \frac{4}{x} + \frac{C}{x^2}$ (f) $y = 3x^4 + Cx^2$

(g) $y = \frac{1}{3}e^{x} + \frac{C}{e^{\frac{1}{2}x}}$ (h) $y = \frac{1}{3}x + \frac{C}{x}$

(i) $y = -x + Cx^2$ (j) $y = \frac{\sin x + C}{x^2}$

(k) $y = 3x + Cx^2$

(3) (a) $\int x \sin x \, dx = \sin x - x\cos x + C$

(b) $y = \frac{\sin x}{x} - \cos x + \frac{C}{x}$

(4) ([illegible]) $y = (x + C)\cos x$

(5) (b) $y = 1 + \frac{C}{\sin x}$

(6) (a) $y = 1 + \frac{2}{e^{x}}$ (b) $y = x + \frac{3}{x^3}$

(c) $y = x^2 - \frac{1}{x^3}$ (d) $y = \frac{1}{x} + \frac{1}{x^2}$

(e) $y = x^2 - x$ (f) $y = x\sin x - x$

(g) $y = (x^2+1)e^{\sin x}$ (h) $y = 2x - \frac{1}{x^2}$

(i) $y = 2 - \frac{1}{e^{x^3}}$ (j) $y = x^4 + \frac{4}{x^2}$

(7) (a) $\int x e^{x} \, dx = xe^{x} - e^{x} + C$

(b) $y = x - 1 + \frac{2}{e^{x}}$

(8) (b)(ii) $y = \sin x + \cos x$

(9) (a) $\int \ln x \, dx = x\ln x - x + C$

(b) $y = \ln x - 1 + \frac{2}{x}$

(10) (b)(ii) $y = (\sin x + \sqrt{2})\cos x$