COMPLEX NUMBERS 1

1. Solve each equation for z.

(a) $z^2+2z+5=0$	(b) $z^2-4z+13=0$	(c) $z^2-2z+17=0$
(d) $z^2+4z+8=0$	(e) $z^2-4z+5=0$	(f) $z^2+9=0$
(g) $z^2-2z+26=0$	(h) $z^2+z+1=0$	(i) $z^2+2z+6=0$

2. Express in the form $x+yi$, where x and y are real numbers:

(a) $(3+4i)+(2+3i)$	(b) $(2-4i)-3(5-3i)$	(c) $(3+5i)+(7-i)$
(d) $(2+7i)-(4-9i)$	(e) $(2+i)(3-4i)$	(f) $(5+4i)(7-i)$
(g) $(3-i)(4-i)$	(h) $(3+4i)(3-4i)$	(i) $(2+i)(3-i)$
(j) $(5-2i)(6+i)$	(k) $(4-3i)(1-i)$	(l) $(3+i)(2-5i)$
(m) $(2+i)(2-i)$	(n) $(2-i)^2$	(o) $i(3+4i)$
(p) $(6+9i)(4-6i)$	(q) $i(1+i)(2+i)$	(r) $(3+2i)^2$
(s) $(2+i)(1-2i)(1+i)$	(t) $(1-3i)^2$	(u) $(2i)^2$

3. Express in the form $x+yi$, where x and y are real numbers:

(a) $(1+2i)^2$ (b) $(1+2i)^3$ (c) $(1+2i)^4$

4. Express in the form $x+yi$, where x and y are real numbers:

(a) $\dfrac{2}{1-i}$	(b) $\dfrac{20}{3+i}$	(c) $\dfrac{4}{1+i}$
(d) $\dfrac{11+2i}{1+2i}$	(e) $\dfrac{14-5i}{3-2i}$	(f) $\dfrac{6+7i}{2-i}$
(g) $\dfrac{7-i}{1+7i}$	(h) $\dfrac{1+i}{1-i}$	(i) $\dfrac{3+i}{4-3i}$
(j) $\dfrac{2i}{1-i}$	(k) $\dfrac{1}{1-2i}$	(l) $\dfrac{5i}{2+i}$
(m) $\dfrac{5}{4-3i}$	(n) $\dfrac{2+3i}{1-i}$	(o) $\dfrac{3-i}{1+2i}$
(p) $\dfrac{3+2i}{3-2i}$	(q) $\dfrac{3+i}{i}$	(r) $\dfrac{-2+3i}{-i}$

5. Express in the form $x+yi$, where x and y are real numbers:

(a) $(2-i)(3+i)$	(b) $(1+2i)(2+3i)$	(c) $(1-i)^3$
(d) $(2+i)(1+i)(3-i)$	(e) $\dfrac{(4-i)(2+3i)}{1-2i}$	(f) $\dfrac{2-i}{(1-i)(1+2i)}$
(g) $\dfrac{1}{(1+2i)^2}$	(h) $\left(\dfrac{2-i}{2+i}\right)^2$	(i) $\dfrac{5-i}{(1+i)^2}$
(j) $\dfrac{(1+3i)(2-i)}{i(3+i)}$		

6. (a) Express $\dfrac{2}{3+i}$ and $\dfrac{3}{2+i}$ in the form $x+yi$, where x and y are real numbers, and hence find $\dfrac{2}{3+i}+\dfrac{3}{2+i}$.

(b) Another way of finding $\dfrac{2}{3+i}+\dfrac{3}{2+i}$ is to use a common denominator of $(3+i)(2+i)$ as follows:

$$\frac{2}{3+i}+\frac{3}{2+i} = \frac{2(2+i)+3(3+i)}{(3+i)(2+i)}$$

Continue the working from here to find $\dfrac{2}{3+i}+\dfrac{3}{2+i}$ in the form $x+yi$, where x and y are real numbers.

7. Simplify $\dfrac{a+bi}{b-ai}$, where a and b are real numbers.

8. Solve each equation for z.

(a) $(1+i)z=3+4i$ (b) $\dfrac{z}{2-3i}=3+i$ (c) $(1-i)z=2+5i$

(d) $\dfrac{z}{2+i}=5-i$

* 9. The worked example below illustrates a method for finding the square roots of a given complex number.

Worked Example

Find the square roots of $5-12i$.

Solution

Let $x+yi$ be a square root of $5-12i$, where x and y are real numbers.

Then $\quad (x+yi)^2 = 5-12i$

$\Rightarrow \quad (x+yi)(x+yi) = 5-12i$

$\Rightarrow \quad x^2+2xyi+y^2i^2 = 5-12i$

$\Rightarrow \quad x^2+2xyi-y^2 = 5-12i$

$\Rightarrow \quad (x^2-y^2)+2xyi = 5-12i$

Equating real parts $\Rightarrow \quad x^2-y^2=5 \quad \ldots\ldots(1)$

Equating imaginary parts $\Rightarrow \quad 2xy=-12 \quad \ldots\ldots(2)$

(2) $\Rightarrow \quad y=\dfrac{-12}{2x}=-\dfrac{6}{x} \quad \ldots\ldots(3)$

Substitute in (1) $\Rightarrow \quad x^2-\left(-\dfrac{6}{x}\right)^2=5$

$\Rightarrow \quad x^2-\dfrac{36}{x^2}=5 \qquad (\times x^2)$

$\Rightarrow \quad x^4-36=5x^2$

$\Rightarrow \quad x^4-5x^2-36=0$

$\Rightarrow \quad (x^2-9)(x^2+4)=0$

$\Rightarrow \quad x^2-9=0 \text{ or } x^2+4=0$

$\Rightarrow \quad x^2=9 \text{ or } x^2=-4$

But x is a real number, so $x^2 \neq -4$.

Hence $x^2=9 \quad \Rightarrow \quad x=\pm 3$

Substitute $x=3$ in (3) $\Rightarrow \quad y=-\dfrac{6}{3}=-2$

Substitute $x=-3$ in (3) $\Rightarrow \quad y=-\dfrac{6}{(-3)}=2$

So the square roots of $5-12i$ are $3-2i$ and $-3+2i$.

[You may verify that $(3-2i)^2=5-12i$ and $(-3+2i)^2=5-12i$.]

Now repeat this method to find the square roots of:

(a) $3+4i$	(b) $15+8i$	(c) $-8+6i$	(d) $3-4i$
(e) $24-10i$	(f) $-7+24i$	(g) $21+20i$	(h) $2i$

ANSWERS

1. (a) $z=-1\pm 2i$ (b) $z=2\pm 3i$ (c) $z=1\pm 4i$
   (d) $z=-2\pm 2i$ (e) $z=2\pm i$ (f) $z=\pm 3i$
   (g) $z=1\pm 5i$ (h) $z=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$ (i) $z=-1\pm\sqrt{5}i$

2. (a) $5+7i$ (b) $-13+5i$ (c) $10+4i$ (d) $-2+16i$
   (e) $10-5i$ (f) $39+23i$ (g) $11-7i$ (h) 25
   (i) $7+i$ (j) $32-7i$ (k) $1-7i$ (l) $11-13i$
   (m) 5 (n) $3-4i$ (o) $-4+3i$ (p) 78
   (q) $-3+i$ (r) $5+12i$ (s) $7+i$ (t) $-8-6i$
   (u) -4

3. (a) $-3+4i$ (b) $-11-2i$ (c) $-7-24i$

4. (a) $1+i$ (b) $6-2i$ (c) $2-2i$ (d) $3-4i$
   (e) $4+i$ (f) $1+4i$ (g) $-i$ (h) i
   (i) $\frac{9}{25}+\frac{13}{25}i$ (j) $-1+i$ (k) $\frac{1}{5}+\frac{2}{5}i$ (l) $1+2i$
   (m) $\frac{4}{5}+\frac{3}{5}i$ (n) $-\frac{1}{2}+\frac{5}{2}i$ (o) $\frac{1}{5}-\frac{7}{5}i$ (p) $\frac{5}{13}+\frac{12}{13}i$
   (q) $1-3i$ (r) $-3-2i$

5. (a) $7+i$ (b) $-4+7i$ (c) $-2-2i$ (d) $6+8i$
   (e) $-\frac{9}{5}+\frac{32}{5}i$ (f) $\frac{1}{2}-\frac{1}{2}i$ (g) $-\frac{3}{25}-\frac{4}{25}i$ (h) $-\frac{7}{25}-\frac{24}{25}i$
   (i) $-\frac{1}{2}-\frac{5}{2}i$ (j) $1-2i$

6. (a) $\frac{2}{3+i}=\frac{3}{5}-\frac{1}{5}i$, $\frac{3}{2+i}=\frac{6}{5}-\frac{3}{5}i$; $\frac{2}{3+i}+\frac{3}{2+i}=\frac{9}{5}-\frac{4}{5}i$
   (b) $\frac{2}{3+i}+\frac{3}{2+i}=\frac{9}{5}-\frac{4}{5}i$

7. $\dfrac{a+bi}{b-ai}=i$

8. (a) $z=\frac{7}{2}+\frac{1}{2}i$ (b) $z=9-7i$ (c) $z=-\frac{3}{2}+\frac{7}{2}i$
   (d) $z=11+3i$

9. (a) $2+i,\ -2-i$ (b) $4+i,\ -4-i$ (c) $1+3i,\ -1-3i$
   (d) $2-i,\ -2+i$ (e) $5-i,\ -5+i$ (f) $3+4i,\ -3-4i$
   (g) $5+2i,\ -5-2i$ (h) $1+i,\ -1-i$

COMPLEX NUMBERS 2

THE MODULUS AND ARGUMENT OF A COMPLEX NUMBER

1. Represent each complex number below on an Argand diagram, then find the modulus and argument (to the nearest $0.1°$ where necessary) of each.

(a) $1+i$	(b) $3+4i$	(c $-5+12i$	(d) $3-2i$
(e) $\sqrt{3}+i$	(f) $-7+24i$	(g) $-4-3i$	(h) $1-i$
(i) $-3+5i$	(j) $1+3i$	(k) $2-2i$	(l) $-4+i$
(m) $4+2i$	(n) $-1+\sqrt{3}i$	(o) $\sqrt{2}(1+i)$	(p) $15-8i$
(q) $-3+\sqrt{3}i$	(r) $5\sqrt{3}+5i$	(s) $2\sqrt{3}+6i$	(t) $-1-i$

2. (a) Noting that $2i = 0+2i$, represent $2i$ on an Argand diagram and then write down the modulus and argument of $2i$ (no calculations should be necessary).
   (b) Noting that $4 = 4+0i$, represent 4 on an Argand diagram and then write down the modulus and argument of 4.
   (c Find the modulus and argument of:
   (i) $-3i$ (ii) -5 (iii) i (iv) $\sqrt{2}i$ (v) 1

3. (a) Express $\dfrac{1-i}{1+i}$ in the form $x+yi$, where x and y are real numbers.
   (b) Hence find the modulus and argument of $\dfrac{1-i}{1+i}$.

4. (a) Express $i(1+i)$ in the form $x+yi$, where x and y are real numbers.
   (b) Hence find the modulus and argument of $i(1+i)$.

5. (a) Express $(1+2i)(3-i)$ in the form $x+yi$, where x and y are real numbers.
   (b) Hence find the modulus and argument of $(1+2i)(3-i)$.

6. (a) Express $\dfrac{7-i}{3-4i}$ in the form $x+yi$, where x and y are real numbers.
   (b) Hence find the modulus and argument of $\dfrac{7-i}{3-4i}$.

7. (a) Express $\dfrac{1+7i}{4+3i}$ in the form $x+yi$, where x and y are real numbers.
   (b) Hence find the modulus and argument of $\dfrac{1+7i}{4+3i}$.

8. (a) Express $\dfrac{(3+i)^2}{1-i}$ in the form $x+yi$, where x and y are real numbers.
   (b) Hence find the modulus and argument (to the nearest $0.1°$) of $\dfrac{(3+i)^2}{1-i}$.

ANSWERS

1. (a) $r = \sqrt{2}$, $\theta = 45°$ (b) $r = 5$, $\theta = 53.1°$
   (c $r = 13$, $\theta = 112.6°$ (d) $r = \sqrt{13}$, $\theta = -33.7°$
   (e) $r = 2$, $\theta = 30°$ (f) $r = 25$, $\theta = 106.3°$
   (g) $r = 5$, $\theta = -143.1°$ (h) $r = \sqrt{2}$, $\theta = -45°$
   (i) $r = \sqrt{34}$, $\theta = 121.0°$ (j) $r = \sqrt{10}$, $\theta = 71.6°$
   (k) $r = 2\sqrt{2}$, $\theta = -45°$ (l) $r = \sqrt{17}$, $\theta = 166.0°$
   (m) $r = 2\sqrt{5}$, $\theta = 26.6°$ (n) $r = 2$, $\theta = 120°$
   (o) $r = 2$, $\theta = 45°$ (p) $r = 17$, $\theta = -28.1°$
   (q) $r = 2\sqrt{3}$, $\theta = 150°$ (r $r = 10$, $\theta = 30°$
   (s) $r = 4\sqrt{3}$, $\theta = 60°$ (t) $r = \sqrt{2}$, $\theta = -135°$

2. (a) $r = 2$, $\theta = 90°$ (b) $r = 4$, $\theta = 0°$
   (c (i) $r = 3$, $\theta = -90°$ (ii) $r = 5$, $\theta = 180°$
   (iii) $r = 1$, $\theta = 90°$ (iv) $r = \sqrt{2}$, $\theta = 90°$
   (v) $r = 1$, $\theta = 0°$

3. (a) $-i$ (b) $r = 1$, $\theta = -90°$

4. (a) $-1 + i$ (b) $r = \sqrt{2}$, $\theta = 135°$

5. (a) $5 + 5i$ (b) $r = 5\sqrt{2}$, $\theta = 45°$

6. (a) $1 + i$ (b) $r = \sqrt{2}$, $\theta = 45°$

7. (a) $1 + i$ (b) $r = \sqrt{2}$, $\theta = 45°$

8. (a) $7 + i$ (b) $r = 5\sqrt{2}$, $\theta = 8.1°$

COMPLEX NUMBERS 3

MULTIPLICATION AND DIVISION OF COMPLEX NUMBERS IN POLAR FORM

1. Let $z = 2(\cos 70° + i\sin 70°)$ and $w = 6(\cos 30° + i\sin 30°)$.
   Express in the form $r(\cos\theta + i\sin\theta)$:

   (a) zw (b) $\dfrac{z}{w}$ (c) z^2 (d) w^2

2. Let $z = 8(\cos 140° + i\sin 140°)$ and $w = 4(\cos 60° + i\sin 60°)$.
   Express in the form $r(\cos\theta + i\sin\theta)$:

   (a) zw (b) $\dfrac{z}{w}$ (c) z^2w^2 (d) $\dfrac{w^3}{z}$

3. Let $z = 4(\cos 30° + i\sin 30°)$ and $w = \sqrt{2}\{\cos(-20°) + i\sin(-20°)\}$.
   Express in the form $r(\cos\theta + i\sin\theta)$:

   (a) zw (b) $\dfrac{z}{w}$ (c) z^2w (d) $\dfrac{z}{w^2}$

4. Let $p = 6(\cos 80° + i\sin 80°)$, $q = 2(\cos 50° + i\sin 50°)$ and $r = 4(\cos 20° + i\sin 20°)$
   Express in the form $r(\cos\theta + i\sin\theta)$:

   (a) pq (b) $\dfrac{p}{q}$ (c) q^2 (d) r^3

   (e) $\dfrac{p}{qr}$ (f) pqr (g) $\dfrac{p^2r}{q^3}$

5. Let $u = 9(\cos 60° + i\sin 60°)$, $v = 3\{\cos(-15°) + i\sin(-15°)\}$ and
   $w = \sqrt{3}(\cos 45° + i\sin 45°)$.
   Express in the form $r(\cos\theta + i\sin\theta)$:

   (a) vw (b) $\dfrac{u}{v}$ (c) w^2 (d) $\dfrac{u^2w}{v^2}$

ANSWERS

1. (a) $12(\cos 100^\circ + i\sin 100^\circ)$ (b) $\frac{1}{3}(\cos 40^\circ + i\sin 40^\circ)$
   (c) $4(\cos 140^\circ + i\sin 140^\circ)$ (d) $36(\cos 60^\circ + i\sin 60^\circ)$

2. (a) $32(\cos 200^\circ + i\sin 200^\circ)$ (b) $2(\cos 80^\circ + i\sin 80^\circ)$
   (c) $1024(\cos 400^\circ + i\sin 400^\circ)$ (d) $8(\cos 40^\circ + i\sin 40^\circ)$

3. (a) $4\sqrt{2}(\cos 10^\circ + i\sin 10^\circ)$ (b) $2\sqrt{2}(\cos 50^\circ + i\sin 50^\circ)$
   (c) $16\sqrt{2}(\cos 40^\circ + i\sin 40^\circ)$ (d) $2(\cos 70^\circ + i\sin 70^\circ)$

4. (a) $12(\cos 130^\circ + i\sin 130^\circ)$ (b) $3(\cos 30^\circ + i\sin 30^\circ)$
   (c) $4(\cos 100^\circ + i\sin 100^\circ)$ (d) $64(\cos 60^\circ + i\sin 60^\circ)$
   (e) $\frac{3}{4}(\cos 10^\circ + i\sin 10^\circ)$ (f) $48(\cos 150^\circ + i\sin 150^\circ)$
   (g) $18(\cos 30^\circ + i\sin 30^\circ)$

5. (a) $3\sqrt{3}(\cos 30^\circ + i\sin 30^\circ)$ (b) $3(\cos 75^\circ + i\sin 75^\circ)$
   (c) $3(\cos 90^\circ + i\sin 90^\circ)$ (d) $9\sqrt{3}(\cos 195^\circ + i\sin 195^\circ)$

COMPLEX NUMBERS 4

DE MOIVRE'S THEOREM

1. Let $z = 1 + \sqrt{3}i$.

   Express z in polar form and hence:
   (a) show that z^6 is a real number
   (b) find z^4 in the form $x + yi$, where x and y are real numbers.

2. Let $z = \sqrt{3} + i$.

   Express z in polar form and hence:
   (a) show that z^3 is pure imaginary (that is, of the form λi for some real number λ)
   (b) find z^5 in the form $x + yi$, where x and y are real numbers.

3. Let $z = 2 + 2\sqrt{3}i$.

   Express z in polar form and hence:
   (a) show that z^3 is a real number
   (b) find z^5 in the form $x + yi$, where x and y are real numbers.

4. Let $z = -1 + \sqrt{3}i$.

   Express z in polar form and hence show that $z^4 - 8z = 0$.

5. Let $z = 1 - i$.

   Express z in polar form and hence find z^7 in the form $x + yi$, where x and y are real numbers.

6. Let $z = 2 + 2i$.

   Express z in polar form and hence show that $z^4 + 64 = 0$.

7. Let $z = \sqrt{3} + i$.

   Express z in polar form and hence show that $z^8 + 16z^4 + 256 = 0$.

8. Let $z = 1 + \sqrt{3}i$.

   Express z in polar form and hence find $\frac{1}{z^5}$ (that is, z^{-5}) in the form $x + yi$, where x and y are real numbers.

9. Let $z = 1 + \sqrt{3}i$ and $w = \sqrt{3} + i$.

   Express z and w in polar form.
   Hence express in the form $x + yi$, where x and y are real numbers:

   (a) $z^4 w^3$ (b) $\frac{z^5}{w^2}$

10. Let $z = -1 + \sqrt{3}i$ and $w = \sqrt{3} + i$.

    Express z and w in polar form.
    Hence express in the form $x + yi$, where x and y are real numbers:

    (a) $z^4 w^5$ (b) $\frac{z^5}{w^3}$

11. Let $z = 1 + \sqrt{3}i$ and $w = \sqrt{3} - i$.

    Express z and w in polar form.
    Hence express in the form $x + yi$, where x and y are real numbers:

    (a) $z^3 w^5$ (b) $\frac{z^4}{w^7}$

12. Let $z = 2\sqrt{3} + 2i$ and $w = 1 + \sqrt{3}i$.

    Express z and w in polar form.
    Hence express in the form $x + yi$, where x and y are real numbers:

    (a) $z^3 w^4$ (b) $\frac{z^4}{w^6}$

13. Let $z = \sqrt{3} + i$ and $w = 1 - i$.

    Express z and w in polar form.
    Hence express in the form $x + yi$, where x and y are real numbers:

    (a) $z^8 w^4$ (b) $\frac{z^5}{w^6}$

14. Let $z = -\sqrt{3} + i$ and $w = 1 - i$.

    Express z and w in polar form.
    Hence express in the form $x + yi$, where x and y are real numbers:

    (a) $z^6 w^3$ (b) $\frac{z^3}{w^4}$

Please turn over for question 15.

✱ 15. Let $a = 1+\sqrt{3}i$, $b = 1+i$ and $c = \sqrt{3}+i$.

Express a, b and c in polar form.
Hence express in the form $x+yi$, where x and y are real numbers:

(a) $\dfrac{a^5b^4}{c^6}$ (b) $\dfrac{a^4b^6}{c^{10}}$ (c) $\dfrac{a^7b^3}{c^8}$ (d) $\dfrac{a^9}{b^2c^4}$ (e) $\dfrac{a^8b^6}{c^7}$ (f) $\dfrac{b^8}{a^5c^2}$

ANSWERS

1. $z = 2(\cos 60° + i\sin 60°)$; (a) $z^6 = 64$ (b) $z^4 = -8-8\sqrt{3}i$ $[= -8(1+\sqrt{3}i)]$

2. $z = 2(\cos 30° + i\sin 30°)$; (a) $z^3 = 8i$ (b) $z^5 = -16\sqrt{3}+16i$ $[= 16(-\sqrt{3}+i)]$

3. $z = 4(\cos 60° + i\sin 60°)$; (a) $z^3 = -64$ (b) $z^5 = 512-512\sqrt{3}i$ $[= 512(1-\sqrt{3}i)]$

4. $z = 2(\cos 120° + i\sin 120°)$

5. $z = \sqrt{2}\{\cos(-45°) + i\sin(-45°)\}$; $z^7 = 8+8i$ $[= 8(1+i)]$

6. $z = 2\sqrt{2}(\cos 45° + i\sin 45°)$

7. $z = 2(\cos 30° + i\sin 30°)$

8. $z = 2(\cos 60° + i\sin 60°)$; $\dfrac{1}{z^5} = \dfrac{1}{64} + \dfrac{\sqrt{3}}{64}i$ $\left[= \dfrac{1}{64}(1+\sqrt{3}i)\right]$

9. $z = 2(\cos 60° + i\sin 60°)$, $w = 2(\cos 30° + i\sin 30°)$;
(a) $z^4w^3 = 64\sqrt{3}-64i$ $[= 64(\sqrt{3}-i)]$
(b) $\dfrac{z^5}{w^2} = -4-4\sqrt{3}i$ $[= -4(1+\sqrt{3}i)]$

10. $z = 2(\cos 120° + i\sin 120°)$, $w = 2(\cos 30° + i\sin 30°)$;
(a) $z^4w^5 = -512i$ (b) $\dfrac{z^5}{w^3} =$ ~~$-4\sqrt{3}+4i$~~ [= ~~$4(-\sqrt{3}+i)$~~]
$= -2\sqrt{3} + 2i$

11. $z = 2(\cos 60° + i\sin 60°)$, $w = 2\{\cos(-30°) + i\sin(-30°)\}$;
(a) $z^3w^5 = 128\sqrt{3}+128i$ $[= 128(\sqrt{3}+i)]$
(b) $\dfrac{z^4}{w^7} = \dfrac{1}{8}i$

12. $z = 4(\cos 30° + i\sin 30°)$, $w = 2(\cos 60° + i\sin 60°)$;
(a) $z^3w^4 = 512\sqrt{3}-512i$ $[= 512(\sqrt{3}-i)]$
(b) $\dfrac{z^4}{w^6} = -2+2\sqrt{3}i$ $[= 2(-1+\sqrt{3}i)]$

13. $z = 2(\cos 30° + i\sin 30°)$, $w = \sqrt{2}\{\cos(-45°) + i\sin(-45°)\}$;
(a) $z^8w^4 = 512+512\sqrt{3}i$ $[= 512(1+\sqrt{3}i)]$
(b) $\dfrac{z^5}{w^6} = 2+2\sqrt{3}i$ $[= 2(1+\sqrt{3}i)]$

14. $z = 2(\cos 150° + i\sin 150°)$, $w = \sqrt{2}\{\cos(-45°) + i\sin(-45°)\}$;
(a) $z^6w^3 = 128+128i$ $[= 128(1+i)]$ (b) $\dfrac{z^3}{w^4} = -2i$

15. $a = 2(\cos 60° + i\sin 60°)$, $b = \sqrt{2}(\cos 45° + i\sin 45°)$, $c = 2(\cos 30° + i\sin 30°)$;
(a) $\dfrac{a^5b^4}{c^6} = 1-\sqrt{3}i$
(b) $\dfrac{a^4b^6}{c^{10}} = -\dfrac{\sqrt{3}}{16} - \dfrac{1}{16}i$ $\left[= -\dfrac{1}{16}(\sqrt{3}+i)\right]$
(c) $\dfrac{a^7b^3}{c^8} = 1-i$
(d) $\dfrac{a^9}{b^2c^4} = 4-4\sqrt{3}i$ $[= 4(1-\sqrt{3}i)]$
(e) $\dfrac{a^8b^6}{c^7} = -16$
(f) $\dfrac{b^8}{a^5c^2} = \dfrac{1}{8}$

ADVANCED HIGHER MATHEMATICS

COMPLEX NUMBERS 5

ROOTS OF COMPLEX NUMBERS

1. Find all the roots of each of these equations, expressing each root in the form $r(\cos\theta + i\sin\theta)$. Show all the roots on a single Argand diagram in each case.

(a) $z^2 = \sqrt{3} + i$	(b) $z^2 = 2 + 2\sqrt{3}i$	(c) $z^2 = -1 + \sqrt{3}i$
(d) $z^2 = 9i$	(e) $z^2 = 2\sqrt{2} + 2\sqrt{2}i$	(f) $z^2 = 8\sqrt{3} - 8i$
(g) $z^2 = -2\sqrt{3} + 2i$	(h) $z^2 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$	(i) $z^2 = -4i$

2. Find all the roots of each of these these equations, expressing each root in the form $r(\cos\theta + i\sin\theta)$. Show all the roots on a single Argand diagram in each case.

(a) $z^3 = 4\sqrt{2} + 4\sqrt{2}i$	(b) $z^3 = 4 + 4\sqrt{3}i$	(c) $z^3 = 8i$
(d) $z^4 = -8 + 8\sqrt{3}i$	(e) $z^4 = -81$	(f) $z^5 = 16\sqrt{3} + 16i$
(g) $z^5 = 16\sqrt{2} - 16\sqrt{2}i$	(h) $z^5 = 32$	(i) $z^6 = 64i$
(j) $z^3 = -32\sqrt{2} + 32\sqrt{2}i$		

3. The <u>cube roots of unity</u> are the three roots of the equation $z^3 = 1$.
Find each of the cube roots of unity in the form $r(\cos\theta + i\sin\theta)$ and show the cube roots of unity on a single Argand diagram.

4. Find each of the <u>cube roots</u> of the complex number $-4\sqrt{3} + 4i$ in the form $r(\cos\theta + i\sin\theta)$ and show the roots on a single Argand diagram.
[Start with the equation $z^3 = -4\sqrt{3} + 4i$.]

5. Find each of the <u>square roots</u> of the complex number $18 + 18\sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$ and show the roots on a single Argand diagram.
[Start with the equation $z^2 = 18 + 18\sqrt{3}i$.]

6. Find each of the <u>fourth roots</u> of -16 in the form $r(\cos\theta + i\sin\theta)$ and show the roots on a single Argand diagram.
[Start with the equation $z^4 = -16$.]

7. Find each of the <u>fifth roots</u> of the complex number, i in the form $r(\cos\theta + i\sin\theta)$ and show the roots on a single Argand diagram.
[Start with the equation $z^5 = i$.]

ANSWERS

1. (a) $z_1 = \sqrt{2}(\cos 15^\circ + i \sin 15^\circ)$, $z_2 = \sqrt{2}(\cos 195^\circ + i \sin 195^\circ)$
   (b) $z_1 = 2(\cos 30^\circ + i \sin 30^\circ)$, $z_2 = 2(\cos 210^\circ + i \sin 210^\circ)$
   (c) $z_1 = \sqrt{2}(\cos 60^\circ + i \sin 60^\circ)$, $z_2 = \sqrt{2}(\cos 240^\circ + i \sin 240^\circ)$
   (d) $z_1 = 3(\cos 45^\circ + i \sin 45^\circ)$, $z_2 = 3(\cos 225^\circ + i \sin 225^\circ)$
   (e) $z_1 = 2(\cos 22.5^\circ + i \sin 22.5^\circ)$, $z_2 = 2(\cos 202.5^\circ + i \sin 202.5^\circ)$
   (f) $z_1 = 4\{\cos(-15^\circ) + i \sin(-15^\circ)\}$, $z_2 = 4(\cos 165^\circ + i \sin 165^\circ)$
   (g) $z_1 = 1(\cos 22.5^\circ + i \sin 22.5^\circ)$, $z_2 = 1(\cos 202.5^\circ + i \sin 202.5^\circ)$
   (h) $z_1 = 2\{\cos(-45^\circ) + i \sin(-45^\circ)\}$, $z_2 = 2(\cos 135^\circ + i \sin 135^\circ)$

2. (a) $z_1 = 2(\cos 15^\circ + i \sin 15^\circ)$, $z_2 = 2(\cos 135^\circ + i \sin 135^\circ)$,
   $z_3 = 2(\cos 255^\circ + i \sin 255^\circ)$
   (b) $z_1 = 2(\cos 20^\circ + i \sin 20^\circ)$, $z_2 = 2(\cos 140^\circ + i \sin 140^\circ)$,
   $z_3 = 2(\cos 260^\circ + i \sin 260^\circ)$
   (c) $z_1 = 2(\cos 30^\circ + i \sin 30^\circ)$, $z_2 = 2(\cos 150^\circ + i \sin 150^\circ)$,
   $z_3 = 2(\cos 270^\circ + i \sin 270^\circ)$
   (d) $z_1 = 2(\cos 30^\circ + i \sin 30^\circ)$, $z_2 = 2(\cos 120^\circ + i \sin 120^\circ)$,
   $z_3 = 2(\cos 210^\circ + i \sin 210^\circ)$, $z_4 = 2(\cos 300^\circ + i \sin 300^\circ)$
   (e) $z_1 = 3(\cos 45^\circ + i \sin 45^\circ)$, $z_2 = 3(\cos 135^\circ + i \sin 135^\circ)$,
   $z_3 = 3(\cos 225^\circ + i \sin 225^\circ)$, $z_4 = 3(\cos 315^\circ + i \sin 315^\circ)$
   (f) $z_1 = 2(\cos 6^\circ + i \sin 6^\circ)$, $z_2 = 2(\cos 78^\circ + i \sin 78^\circ)$,
   $z_3 = 2(\cos 150^\circ + i \sin 150^\circ)$, $z_4 = 2(\cos 222^\circ + i \sin 222^\circ)$,
   $z_5 = 2(\cos 294^\circ + i \sin 294^\circ)$
   (g) $z_1 = 2(\cos 9^\circ + i \sin 9^\circ)$, $z_2 = 2(\cos 81^\circ + i \sin 81^\circ)$,
   $z_3 = 2(\cos 153^\circ + i \sin 153^\circ)$, $z_4 = 2(\cos 225^\circ + i \sin 225^\circ)$,
   $z_5 = 2(\cos 297^\circ + i \sin 297^\circ)$
   (h) $z_1 = 2(\cos 0^\circ + i \sin 0^\circ)$, $z_2 = 2(\cos 72^\circ + i \sin 72^\circ)$,
   $z_3 = 2(\cos 144^\circ + i \sin 144^\circ)$, $z_4 = 2(\cos 216^\circ + i \sin 216^\circ)$,
   $z_5 = 2(\cos 288^\circ + i \sin 288^\circ)$
   (i) $z_1 = 2(\cos 15^\circ + i \sin 15^\circ)$, $z_2 = 2(\cos 75^\circ + i \sin 75^\circ)$,
   $z_3 = 2(\cos 135^\circ + i \sin 135^\circ)$, $z_4 = 2(\cos 195^\circ + i \sin 195^\circ)$,
   $z_5 = 2(\cos 255^\circ + i \sin 255^\circ)$, $z_6 = 2(\cos 315^\circ + i \sin 315^\circ)$
   (j) $z_1 = 4(\cos 45^\circ + i \sin 45^\circ)$, $z_2 = 4(\cos 165^\circ + i \sin 165^\circ)$,
   $z_3 = 4(\cos 285^\circ + i \sin 285^\circ)$

3. $z_1 = 1(\cos 0^\circ + i \sin 0^\circ)$, $z_2 = 1(\cos 120^\circ + i \sin 120^\circ)$, $z_3 = 1(\cos 240^\circ + i \sin 240^\circ)$
4. $z_1 = 2(\cos 50^\circ + i \sin 50^\circ)$, $z_2 = 2(\cos 170^\circ + i \sin 170^\circ)$, $z_3 = 2(\cos 290^\circ + i \sin 290^\circ)$
5. $z_1 = 6(\cos 30^\circ + i \sin 30^\circ)$, $z_2 = 6(\cos 210^\circ + i \sin 210^\circ)$
6. $z_1 = 2(\cos 45^\circ + i \sin 45^\circ)$, $z_2 = 2(\cos 135^\circ + i \sin 135^\circ)$, $z_3 = 2(\cos 225^\circ + i \sin 225^\circ)$,
   $z_4 = 2(\cos 315^\circ + i \sin 315^\circ)$
7. $z_1 = 1(\cos 18^\circ + i \sin 18^\circ)$, $z_2 = 1(\cos 90^\circ + i \sin 90^\circ)$, $z_3 = 1(\cos 162^\circ + i \sin 162^\circ)$,
   $z_4 = 1(\cos 234^\circ + i \sin 234^\circ)$, $z_5 = 1(\cos 306^\circ + i \sin 306^\circ)$

COMPLEX NUMBERS 6

POLYNOMIAL EQUATIONS

1. Show that $z = 2$ is a root of the equation $z^3 - 4z^2 + 6z - 4 = 0$ and hence solve this equation completely.

2. Show that $z = 3$ is a root of the equation $z^3 - 5z^2 + 11z - 15 = 0$ and hence solve this equation completely.

3. Show that $z = 1$ is a root of the equation $z^3 - 5z^2 + 17z - 13 = 0$ and hence solve this equation completely.

4. Show that $z = -2$ is a root of the equation $z^3 - 4z^2 - 2z + 20 = 0$ and hence solve this equation completely.

5. Show that $z = 4$ is a root of the equation $z^3 - 2z^2 - 6z - 8 = 0$ and hence solve this equation completely.

6. Show that $z = -1$ is a root of the equation $z^3 + 3z^2 + 12z + 10 = 0$ and hence solve this equation completely.

7. Show that $z = 2$ is a root of the equation $z^3 + 4z^2 + 13z - 50 = 0$ and hence solve this equation completely.

8. Show that $z = 1$ is a root of the equation $z^3 - 5z^2 + 12z - 8 = 0$ and hence solve this equation completely.

9. Show that $z = -2$ is a root of the equation $z^3 + z + 10 = 0$ and hence solve this equation completely.

10. Find a real root of the equation $z^3 + 5z^2 - 4z - 60 = 0$ and hence solve this equation completely.

11. Find a real root of the equation $z^3 + 2z^2 - 3z - 10 = 0$ and hence solve this equation completely.

12. Find a real root of the equation $z^3 + 2z^2 + 9z + 18 = 0$ and hence solve this equation completely.

13. Find a real root of the equation $z^3 - 7z^2 + 24z - 18 = 0$ and hence solve this equation completely.

14. <u>Given</u> that $z = 1 + i$ is a root of the equation $z^4 - 4z^3 + 11z^2 - 14z + 10 = 0$, write down another root and hence solve this equation completely.

15. <u>Given</u> that $z = 2 + i$ is a root of the equation $z^4 - 10z^3 + 42z^2 - 82z + 65 = 0$, write down another root and hence solve this equation completely.

16. <u>Given</u> that $z = 2 + 3i$ is a root of the equation $z^4 - 2z^3 + 10z^2 + 6z + 65 = 0$, write down another root and hence solve this equation completely.

17. <u>Given</u> that $z = 3 - 2i$ is a root of the equation $z^4 - 8z^3 + 27z^2 - 38z + 26 = 0$, write down another root and hence solve this equation completely.

18. <u>Given</u> that $z = 1 + 3i$ is a root of the equation $z^4 - 6z^3 + 23z^2 - 50z + 50 = 0$, write down another root and hence solve this equation completely.

19. <u>Given</u> that $z = 2 + 3i$ is a root of the equation $z^4 - 6z^3 + 23z^2 - 34z + 26 = 0$, write down another root and hence solve this equation completely.

20. <u>Given</u> that $z = -1 + i$ is a root of the equation $z^4 + 8z^3 + 24z^2 + 32z + 20 = 0$, write down another root and hence solve this equation completely.

21. <u>Given</u> that $z = -2 + i$ is a root of the equation $z^4 + 6z^3 + 10z^2 - 2z - 15 = 0$, write down another root and hence solve this equation completely.

22. <u>Given</u> that $z = 1 - 3i$ is a root of the equation $z^4 - 3z^3 + 10z^2 - 6z - 20 = 0$, write down another root and hence solve this equation completely.

23. (a) Show that $z = 1 + i$ is a root of the equation $z^4 - 2z^3 + 6z^2 - 8z + 8 = 0$.
(b) Write down another root.
(c) Hence solve the equation completely.

24. (a) Show that $z = 2 - i$ is a root of the equation $z^4 - 2z^3 - z^2 + 2z + 10 = 0$.
(b) Write down another root.
(c) Hence solve the equation completely.

25. (a) Show that $z = 1 + 2i$ is a root of the equation $z^3 + z^2 - z + 15 = 0$.
(b) Write down another root.
(c) Hence solve the equation completely.

26. (a) Show that $z = 1 + 3i$ is a root of the equation $z^4 + 2z^3 + 7z^2 + 30z + 50 = 0$.
(b) Write down another root.
(c) Hence solve the equation completely.

27. (a) Show that $z = 1 - i$ is a root of the equation $z^4 + 2z^3 + 2z^2 - 8z + 16 = 0$.
(b) Write down another root.
(c) Hence solve the equation completely.

28. (a) Show that $z = 1-2i$ is a root of the equation $z^4 - 3z^3 + z^2 + 7z - 30 = 0$.
(b) Write down another root.
(c) Hence solve the equation completely.

29. (a) Show that $z = -1+i$ is a root of the equation $z^4 + 8z^2 + 16z + 20 = 0$.
(b) Write down another root.
(c) Hence solve the equation completely.

30. (a) Show that $z = 1+i$ is a root of the equation $z^4 + 4z^3 - 8z + 20 = 0$.
(b) Write down another root.
(c) Hence solve the equation completely.

31. (a) Show that $z = 2+i$ is a root of the equation $z^3 - z^2 - 7z + 15 = 0$.
(b) Write down another root.
(c) Hence solve the equation completely.

32. (a) Show that $z = 1+i$ is a root of the equation $z^4 - z^3 - z^2 + 4z - 2 = 0$.
(b) Write down another root.
(c) Hence solve the equation completely.

33. Find two distinct real roots of the equation $z^4 - 3z^3 + 5z^2 - z - 10 = 0$, and hence solve this equation completely.

34. Find two distinct real roots of the equation $z^4 - 7z^3 + 27z^2 - 47z + 26 = 0$, and hence solve this equation completely.

ANSWERS

1. $z = 2$ or $z = 1 \pm i$
2. $z = 3$ or $z = 1 \pm 2i$
3. $z = 1$ or $z = 2 \pm 3i$
4. $z = -2$ or $z = 3 \pm i$
5. $z = 4$ or $z = -1 \pm i$
6. $z = -1$ or $z = -1 \pm 3i$
7. $z = 2$ or $z = -3 \pm 4i$
8. $z = 1$ or $z = 2 \pm 2i$
9. $z = -2$ or $z = 1 \pm 2i$
10. $z = 3$ or $z = -4 \pm 2i$
11. $z = 2$ or $z = -2 \pm i$
12. $z = -2$ or $z = \pm 3i$
13. $z = 1$ or $z = 3 \pm 3i$
14. $z = 1 \pm i$ or $z = 1 \pm 2i$
15. $z = 2 \pm i$ or $z = 3 \pm 2i$
16. $z = 2 \pm 3i$ or $z = -1 \pm 2i$
17. $z = 3 \pm 2i$ or $z = 1 \pm i$
18. $z = 1 \pm 3i$ or $z = 2 \pm i$
19. $z = 2 \pm 3i$ or $z = 1 \pm i$
20. $z = -1 \pm i$ or $z = -3 \pm i$
21. $z = -2 \pm i$, $z = 1$ or $z = -3$
22. $z = 1 \pm 3i$, $z = 2$ or $z = -1$

23. (b) $z = 1-i$ (c) $z = 1 \pm i$ or $z = \pm 2i$
24. (b) $z = 2+i$ (c) $z = 2 \pm i$ or $z = -1 \pm i$
25. (b) $z = 1-2i$ (c) $z = 1 \pm 2i$ or $z = -3$
26. (b) $z = 1-3i$ (c) $z = 1 \pm 3i$ or $z = -2 \pm i$
27. (b) $z = 1+i$ (c) $z = 1 \pm i$ or $z = -2 \pm 2i$
28. (b) $z = 1+2i$ (c) $z = 1 \pm 2i$, $z = 3$ or $z = -2$
29. (b) $z = -1-i$ (c) $z = -1 \pm i$ or $z = 1 \pm 3i$
30. (b) $z = 1-i$ (c) $z = 1 \pm i$ or $z = -3 \pm i$
31. (b) $z = 2-i$ (c) $z = 2 \pm i$ or $z = -3$
32. (b) $z = 1-i$ (c) $z = 1 \pm i$ or $z = \dfrac{-1 \pm \sqrt{5}}{2}$

33. $z = 2$, $z = -1$ or $z = 1 \pm 2i$
34. $z = 1$, $z = 2$ or $z = 2 \pm 3i$

COMPLEX NUMBERS 7

TRIGONOMETRIC IDENTITIES

1. Write down the binomial expansion of $(a+b)^2$.

   Starting with $\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2$, prove that:

   (i) $\cos 2\theta = \cos^2\theta - \sin^2\theta$
   (ii) $\sin 2\theta = 2\sin\theta\cos\theta$

   Hence find an identity for $\tan 2\theta$ in terms of $\tan\theta$.

2. Write down the binomial expansion of $(a+b)^3$.

   Starting with $\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$, prove that:

   (i) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$
   (ii) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

   Hence find an identity for $\tan 3\theta$ in terms of $\tan\theta$.

3. Write down the binomial expansion of $(a+b)^4$.

   Starting with $\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$, prove that:

   (i) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$
   (ii) $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

   Hence find an identity for $\tan 4\theta$ in terms of $\tan\theta$.

   Also, make use of the identity $\sin^2\theta + \cos^2\theta = 1$ to find an identity for $\cos 4\theta$ entirely in terms of $\cos\theta$.

✱ 4. Prove that

$$\tan 6\theta = 2\tan\theta\left(\frac{3 - 10\tan^2\theta + 3\tan^4\theta}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}\right).$$

ANSWERS

1. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

2. $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

3. $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$; $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

ADVANCED HIGHER MATHEMATICS

COMPLEX NUMBERS 8

FURTHER TRIGONOMETRIC IDENTITIES

You may assume the following results for the complex number $z = \cos\theta + i\sin\theta$:

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

$$z^n - \frac{1}{z^n} = 2i\sin n\theta$$

1. Starting with $(2\cos\theta)^2 = \left(z + \frac{1}{z}\right)^2$,

show that $\cos^2\theta = \frac{1}{2}\cos 2\theta + \frac{1}{2}$.

Hence find $\int \cos^2\theta \, d\theta$.

2. Starting with $(2\cos\theta)^3 = \left(z + \frac{1}{z}\right)^3$,

show that $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$.

Hence find $\int \cos^3\theta \, d\theta$.

3. Starting with $(2\cos\theta)^5 = \left(z + \frac{1}{z}\right)^5$,

show that $\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$.

Hence find $\int \cos^5\theta \, d\theta$.

4. Starting with $(2\cos\theta)^6 = \left(z + \frac{1}{z}\right)^6$,

show that $\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$.

Hence find $\int \cos^6\theta \, d\theta$.

5. Starting with $(2i\sin\theta)^2 = \left(z - \frac{1}{z}\right)^2$,

show that $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$.

Hence find $\int \sin^2\theta \, d\theta$.

6. Starting with $(2i\sin\theta)^4 = \left(z - \frac{1}{z}\right)^4$,

show that $\sin^4\theta = \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}$.

Hence find $\int \sin^4\theta \, d\theta$.

7. Starting with $(2i\sin\theta)^5 = \left(z - \frac{1}{z}\right)^5$,

show that $\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$.

Hence find $\int \sin^5\theta \, d\theta$.

8. Starting with $(2i\sin\theta)^6 = \left(z - \frac{1}{z}\right)^6$,

show that $\sin^6\theta = \frac{5}{16} - \frac{15}{32}\cos 2\theta + \frac{3}{16}\cos 4\theta - \frac{1}{32}\cos 6\theta$.

Hence find $\int \sin^6\theta \, d\theta$.

Please turn over for answers.

ANSWERS

1. $\int \cos^2 \theta \, d\theta = \frac{1}{4}\sin 2\theta + \frac{1}{2}\theta + C$

2. $\int \cos^3 \theta \, d\theta = \frac{1}{12}\sin 3\theta + \frac{3}{4}\sin \theta + C$

3. $\int \cos^5 \theta \, d\theta = \frac{1}{80}\sin 5\theta + \frac{5}{48}\sin 3\theta + \frac{5}{8}\sin \theta + C$

4. $\int \cos^6 \theta \, d\theta = \frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta + C$

5. $\int \sin^2 \theta \, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$

6. $\int \sin^4 \theta \, d\theta = \frac{1}{32}\sin 4\theta - \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta + C$

7. $\int \sin^5 \theta \, d\theta = -\frac{1}{80}\cos 5\theta + \frac{5}{48}\cos 3\theta - \frac{5}{8}\cos \theta + C$

8. $\int \sin^6 \theta \, d\theta = \frac{5}{16}\theta - \frac{15}{64}\sin 2\theta + \frac{3}{64}\sin 4\theta - \frac{1}{192}\sin 6\theta + C$

COMPLEX NUMBERS 9

LOCUS IN THE COMPLEX PLANE

1. The complex number z moves in the complex plane subject to each condition given below. Find the equation of the locus of z and sketch the locus in each case.

(a)	$\lvert z\rvert = 5$	(b)	$\lvert z-3\rvert = 2$	(c)	$\lvert z+1\rvert = 4$
(d)	$\lvert z-2i\rvert = 1$	(e)	$\lvert z+i\rvert = 3$	(f)	$\lvert z-1\rvert = 6$
(g)	$\lvert z-3i\rvert = 2$	(h)	$\lvert z+2i\rvert = 5$	(i)	$\lvert z-i\rvert = 4$
(j)	$\lvert z-1-2i\rvert = 3$	(k)	$\lvert z+1-2i\rvert = 5$	(l)	$\lvert z\rvert \le 2$
(m)	$\lvert z-3\rvert \le 1$	(n)	$\lvert z-i\rvert < 4$	(o)	$\lvert z+1\rvert > 2$
(p)	$\arg(z) = \frac{\pi}{4}$	(q)	$\arg(z) = \frac{\pi}{6}$		

2. The complex number z moves in the complex plane such that $\lvert z-1\rvert = \lvert z-i\rvert$.
Show that the locus of z is the straight line with equation $y = x$.

3. The complex number z moves in the complex plane such that $\lvert z-2\rvert = \lvert z-i\rvert$.
Show that the locus of z is the straight line with equation $2y = 4x - 3$.

4. The complex number z moves in the complex plane such that $\lvert z-3\rvert = \lvert z-2i\rvert$.
Show that the locus of z is the straight line with equation $4y = 6x - 5$.

ANSWERS

1. (a) $x^2 + y^2 = 25$; the locus of z is the circumference of the circle with centre O and radius 5

(b) $(x-3)^2 + y^2 = 4$; the locus of z is the circumference of the circle with centre (3, 0) and radius 2

(c) $(x+1)^2 + y^2 = 16$; the locus of z is the circumference of the circle with centre (–1, 0) and radius 4

(d) $x^2 + (y-2)^2 = 1$; the locus of z is the circumference of the circle with centre (0, 2) and radius 1

(e) $x^2 + (y+1)^2 = 9$; the locus of z is the circumference of the circle with centre (0, –1) and radius 3

(f) $(x-1)^2 + y^2 = 36$; the locus of z is the circumference of the circle with centre (1, 0) and radius 6

(g) $x^2 + (y-3)^2 = 4$; the locus of z is the circumference of the circle with centre (0, 3) and radius 2

(h) $x^2 + (y+2)^2 = 25$; the locus of z is the circumference of the circle with centre (0, –2) and radius 5

(i) $x^2 + (y-1)^2 = 16$; the locus of z is the circumference of the circle with centre (0, 1) and radius 4

(j) $(x-1)^2 + (y-2)^2 = 9$; the locus of z is the circumference of the circle with centre (1, 2) and radius 3

(k) $(x+1)^2 + (y-2)^2 = 25$; the locus of z is the circumference of the circle with centre (–1, 2) and radius 5

(l) $x^2 + y^2 \le 4$; the locus of z is the circumference and inside of the circle with centre O and radius 2

(m) $(x-3)^2 + y^2 \le 1$; the locus of z is the circumference and inside of the circle with centre (3, 0) and radius 1

(n) $x^2 + (y-1)^2 < 16$; the locus of z is the inside of the circle with centre (0, 1) and radius 4

(o) $(x+1)^2 + y^2 > 4$; the locus of z is the set of points outside the circle with centre (–1, 0) and radius 2

(p) $y = x$, $x > 0$; the locus of z is the portion of the straight line with equation $y = x$ such that $x > 0$

(q) $y = \frac{x}{\sqrt{3}}$, $x > 0$; the locus of z is the portion of the straight line with equation $y = \frac{x}{\sqrt{3}}$ such that $x > 0$

<u>COMPLEX NUMBERS HOMEWORK 1</u>

1. Solve the equation $z^2 + 10z + 29 = 0$ for the complex number z.

2. Express in the form $x + yi$, where x and y are real numbers:

   (a) $(3+2i)(1+4i)$ (b) $\dfrac{7+26i}{5+2i}$ (c) $\dfrac{1+8i}{5+i}$

   (d) $\dfrac{9+7i}{(1+i)(2+i)}$ (e) $\dfrac{8-i}{1-2i} + \dfrac{7+6i}{2+i}$

3. Solve each equation below for the complex number z.

   (a) $(3+i)z = -1+i$ (b) $\dfrac{z}{2-5i} = 2+i$

4. Find the modulus and argument of the complex number $-2\sqrt{3}+6i$.

5. Find the two roots of the equation $z^2 - 6z + 16 = 0$, expressing each solution in the form $x + yi$, where x and y are real numbers.

   Find the modulus and argument (correct to the nearest $0.1°$) of each root.

6. Express the complex number $\dfrac{4}{1-i}$ in the form $x + yi$, where x and y are real numbers, and hence find the modulus and argument of $\dfrac{4}{1-i}$.

7. Let $z = -3 + \sqrt{3}i$.

   Express z in polar form and hence, or otherwise, find z^4 in the form $x + yi$, where x and y are real numbers.

8. Let $z = 1 + \sqrt{3}i$ and $w = 1 + i$.

   (a) Find the modulus and argument of z, and hence express z^5 in the form $r(\cos\theta + i\sin\theta)$ for some values of r and θ.

   (b) Find the modulus and argument of w, and hence express w^6 in the form $r(\cos\theta + i\sin\theta)$ for some values of r and θ.

   (c) Using your expressions in (a) and (b) above, find in the form $x + yi$, where x and y are real numbers:

   (i) $z^5 w^6$ (ii) $\dfrac{z^5}{w^6}$