EQUATIONS OF A STRAIGHT LINE IN 3-DIMENSIONS

1. Find the equation of each line in **symmetric form**.

(a) A line passes through the point P(1, –2, 3) and is parallel to the vector $2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

(b) A line passes though the point P(–1, 2, –2) and is parallel to the vector $\mathbf{i} - \mathbf{j} + \mathbf{k}$.

(c) A line passes through the point P(4, 2, –1) and is parallel to the vector $3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

(d) A line passes through the point P(1, –1, 1) and is parallel to the vector $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

(e) A line passes through the point P(3, 4, 5) and is parallel to the vector $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.

(f) A line passes through the point P(–1, 3, 0) and is parallel to the vector $-2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$.

(g) A line passes through the point P(0, –1, 2) and is parallel to the vector $-\mathbf{i} + \mathbf{j} - \mathbf{k}$.

(h) A line passes through the point P(2, –2, –2) and is parallel to the vector $-\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

2. Find the equation of the line AB in **symmetric form** in each case.

(a) A(0, 1, 3) and B(–1, 2, –4) (b) A(5, –1, 0) and B(6, 2, –7)

(c) A(3, 11, –2) and B(6, –1, 0) (d) A(2, 1, 0) and B(3, 7, 10)

(e) A(0, 0, 0) and B(1, 2, 3) (f) A(1, –2, –1) and B(2, 3, 1)

(g) A(3, –1, 6) and B(0, –3, –1) (h) A(1, 2, –1) and B(–1, 0, 1)

3. Triangle ABC has vertices A(2, –1, 3), B(4, 3, 5) and C(–3, 2, 1).

Find, in **symmetric form**, the equations of the sides AB, BC and AC.

4. Find the equation of each line in **parametric form**.

(a) A line passes through the point P(2, 2, 1) and is parallel to the vector $3\mathbf{i} - \mathbf{j} - \mathbf{k}$.

(b) A line passes through the point P(2, 1, –1) and is parallel to the vector $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

(c) A line passes through the point P(2, –1, 6) and is parallel to the vector $\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}$.

(d) A line passes through the point P(3, 4, 5) and is parallel to the vector $2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$.

(e) A line passes through the point P(3, –1, 5) and is parallel to the vector $-2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

(f) A line passes through the point P(–1, 0, 2) and is parallel to the vector $3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

(g) A line passes through the point P(2, –3, 1) and is parallel to the vector $-\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

(h) A line passes through the point P(1, –1, 1) and is parallel to the vector $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

(i) A line passes through the point P(2, 1, –3) and is parallel to the vector $3\mathbf{i} - \mathbf{j} - \mathbf{k}$.

(j) A line passes through the point P(3, 4, 5) and is parallel to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

(k) A line passes through the points A(2, 1, 3) and B(3, 4, 5).

(l) A line passes through the points A(–1, 2, 0) and B(2, 3, –1).

(m) A line passes through the points A(1, –2, –1) and B(2, 3, 1).

5. A line has parametric equations

$$x = -2t + 2, \quad y = 3t - 1, \quad z = -t.$$

Find the equation of this line in symmetric form.

6. The equation of a line in symmetric form is

$$\frac{x+3}{2} = \frac{y-1}{1} = \frac{z-4}{-3}.$$

Find the parametric equations of this line.

7. The equation of a line in symmetric form is

$$\frac{x+4}{1}=\frac{y-3}{-2}=\frac{z-4}{2}.$$

Find the parametric equations of this line.

8. A line has parametric equations

$$x=3+2t, \quad y=-1+3t, \quad z=1+t.$$

Find the equation of this line in symmetric form.

9. A line has parametric equations

$$x=3+4t, \quad y=-2-3t, \quad z=4-t.$$

Find the equation of this line in symmetric form.

10. Show that the lines L_1 and L_2 intersect in each case and find the coordinates of the point of intersection.

(a) L_1: $\frac{x-1}{1}=\frac{y+1}{1}=\frac{z-2}{1}$

L_2: $\frac{x-2}{1}=\frac{y+2}{3}=\frac{z+1}{5}$

(b) L_1: $\frac{x-2}{1}=\frac{y-2}{3}=\frac{z-3}{1}$

L_2: $\frac{x-2}{1}=\frac{y-3}{4}=\frac{z-4}{2}$

(c) L_1: $x=-2+2t, \quad y=1-3t, \quad z=-1+t$

L_2: $\frac{x+3}{-1}=\frac{y-4}{1}=\frac{z}{-1}$

(d) L_1: $\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}$

L_2: $\frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3}$

(e) L_1: $\frac{x-1}{2}=\frac{y-3}{3}=\frac{z-2}{1}$

L_2: $\frac{x+1}{2}=\frac{y-6}{1}=\frac{z-7}{-1}$

11. The equations of lines L_1 and L_2 are given below.

L_1: $\frac{x-4}{3}=\frac{y}{-1}=\frac{z-2}{-1}$

L_2: $\frac{x}{1}=\frac{y}{1}=\frac{z-3}{1}$

Show that lines L_1 and L_2 do **not** intersect.

12. Calculate the **acute** angle between the lines L_1 and L_2 in each part of question 10.

Questions 13, 14 and 15 are miscellaneous questions.

13. Obtain the parametric equations of the line passing through the points A(2, –3, 1) and B(1, –1, 7).

14. A line passes through the points A(2, 1, 1) and B(4, 5, 6).

(a) Find the equation of the line AB in symmetric form.
(b) Write down the parametric equations of the line AB.

15. The equations of lines L_1 and L_2 are given below.

L_1: $\frac{x+4}{3}=\frac{y+7}{5}=\frac{z+12}{8}$

L_2: $\frac{x+3}{1}=\frac{y}{-1}=\frac{z+10}{3}$

(a) Show that the lines L_1 and L_2 intersect and find the coordinates of the point of intersection.

(b) Calculate the acute angle between the lines L_1 and L_2.

ANSWERS

1. (a) $\frac{x-1}{2}=\frac{y+2}{1}=\frac{z-3}{-1}$ (b) $\frac{x+1}{1}=\frac{y-2}{-1}=\frac{z+2}{1}$

(c) $\frac{x-4}{3}=\frac{y-2}{1}=\frac{z+1}{3}$ (d) $\frac{x-1}{2}=\frac{y+1}{2}=\frac{z-1}{1}$

(e) $\frac{x-3}{2}=\frac{y-4}{1}=\frac{z-5}{-3}$ (f) $\frac{x+1}{-2}=\frac{y-3}{-1}=\frac{z}{4}$

(g) $\frac{x}{-1}=\frac{y+1}{1}=\frac{z-2}{-1}$ (h) $\frac{x-2}{-1}=\frac{y+2}{3}=\frac{z+2}{-1}$

2. (a) $\frac{x}{-1}=\frac{y-1}{1}=\frac{z-3}{-7}$ (b) $\frac{x-5}{1}=\frac{y+1}{3}=\frac{z}{-7}$

(c) $\frac{x-3}{3}=\frac{y-11}{-12}=\frac{z+2}{2}$ (d) $\frac{x-2}{1}=\frac{y-1}{6}=\frac{z}{10}$

(e) $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ (f) $\frac{x-1}{1}=\frac{y+2}{5}=\frac{z+1}{2}$

(g) $\frac{x-3}{-3}=\frac{y+1}{-2}=\frac{z-6}{-7}$ (h) $\frac{x-1}{-2}=\frac{y-2}{-2}=\frac{z+1}{2}$

3. AB: $\frac{x-2}{2}=\frac{y+1}{4}=\frac{z-3}{2}$ BC: $\frac{x-4}{-7}=\frac{y-3}{-1}=\frac{z-5}{-4}$

AC: $\frac{x-2}{-5}=\frac{y+1}{3}=\frac{z-3}{-2}$

4. (a) $x=3t+2,\ y=-t+2,\ z=-t+1$ (b) $x=2t+2,\ y=t+1,\ z=3t-1$

(c) $x=t+2,\ y=2t-1,\ z=-8t+6$ (d) $x=2t+3,\ y=-3t+4,\ z=-4t+5$

(e) $x=-2t+3,\ y=4t-1,\ z=-t+5$ (f) $x=3t-1,\ y=t,\ z=2t+2$

(g) $x=-t+2,\ y=-2t-3,\ z=t+1$ (h) $x=3t+1,\ y=2t-1,\ z=t+1$

(i) $x=3t+2,\ y=-t+1,\ z=-t-3$ (j) $x=t+3,\ y=t+4,\ z=t+5$

(k) $x=t+2,\ y=3t+1,\ z=2t+3$ (l) $x=3t-1,\ y=t+2,\ z=-t$

(m) $x=t+1,\ y=5t-2,\ z=2t-1$

5. $\frac{x-2}{-2}=\frac{y+1}{3}=\frac{z}{-1}$

6. $x=2t-3,\ y=t+1,\ z=-3t+4$

7. $x=t-4,\ y=-2t+3,\ z=2t+4$

8. $\frac{x-3}{2}=\frac{y+1}{3}=\frac{z-1}{1}$

9. $\frac{x-3}{4}=\frac{y+2}{-3}=\frac{z-4}{-1}$

10. (a) (3, 1 4) (b) (1, −1, 2) (c) (−6, 7, −3) (d) (1, 3, 2)
(e) (5, 9, 4)

12. (a) 28·6° (b) 9·3° (c) 22·2° (d) 73·0°
(e) 49·1°

13. $x=-t+2,\ y=2t-3,\ z=6t+1$

14. (a) $\frac{x-2}{2}=\frac{y-1}{4}=\frac{z-1}{5}$ (b) $x=2t+2,\ y=4t+1,\ z=5t+1$

15. (a) (−1, −2, −4) (b) 47·9°

THE EQUATION OF A PLANE

1. (a) Find the equation of the plane perpendicular to the vector $2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and containing the point P(0, 2, 6).

   (b) Find the equation of the plane perpendicular to the vector $5\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and containing the point P(2, 1, –1).

   (c) Find the equation of the plane perpendicular to the vector $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and containing the point P(5, 3, –2).

   (d) Find the equation of the plane perpendicular to the vector $-4\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}$ and containing the point P(–4, 6, 7).

   (e) Find the equation of the plane perpendicular to the vector $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and containing the point P(–1, 2, 1).

   (f) Find the equation of the plane perpendicular to the vector $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and containing the point P(1, –3, 1).

2. Find in each case the equation of the plane perpendicular to PQ which contains the point P.

   (a) P(0, 1, 4) and Q(1, 2, 7) (b) P(3, –2, 1) and Q(5, –7, 3)
   (c) P(5, –1, 0) and Q(2, 2, –5) (d) P(–7, 3, 3) and Q(1, 1, 4)

3. The equation of a line L is given by $\dfrac{x-1}{2} = \dfrac{y+4}{-1} = \dfrac{z-2}{3}$.
   The plane π is perpendicular to the line L and contains the point (1, 1, 2).

   (a) Write down the components of a vector normal to the plane π.
   (b) Find the equation of the plane π.

4. A plane is parallel to each of the vectors $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $4\mathbf{i} - 2\mathbf{k}$.

   (a) Find a vector normal to this plane.
   (b) Given that the plane contains the point (1, 1, 0), find the equation of the plane.

5. A plane is parallel to each of the vectors $4\mathbf{i} - \mathbf{k}$ and $6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

   (a) Find a vector normal to this plane.
   (b) Given that the plane contains the point (3, 4, –7), find the equation of the plane.

6. A plane is parallel to each of the vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $-2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$.

   (a) Find a vector normal to this plane.
   (b) Given that the plane contains the origin, find the equation of the plane.

7. A plane is parallel to each of the vectors $\mathbf{i} - \mathbf{k}$ and $6\mathbf{j} + 5\mathbf{k}$.

   (a) Find a vector normal to this plane.
   (b) Given that the plane contains the point (–2, 3, 7), find the equation of the plane.

8. A plane is parallel to each of the vectors $-2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $-\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.

   (a) Find a vector normal to this plane.
   (b) Given that the plane contains the point (2, 3, –1), find the equation of the plane.

9. A plane is parallel to each of the vectors $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $2\mathbf{j} - \mathbf{k}$.

   (a) Find a vector normal to this plane.
   (b) Given that the plane contains the point (1, 2, –1), find the equation of the plane.

10. A plane contains the points A(1, 2, 1), B(–1, 0, 3) and C(0, 5, –1).

    (a) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in component form and hence find a vector normal to the plane.
    (b) Find the equation of the plane.

11. A plane contains the points O(0, 0, 0), A(1, 2, 1) and B(–2, 1, 2).

    (a) Find the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in component form and hence find a vector normal to the plane.
    (b) Find the equation of the plane.

12. Using a similar method to that of questions 10 and 11, find the equation of the plane containing the points:

    (a) A(2, 1, 2), B(0, 3, –1) and C(3, 0, 4)
    (b) A(–1, 1, 0), B(3, 3, 3) and C(2, –1, 2)
    (c) A(–1, 3, 1), B(1, –3, –3) and C(3, –1, 5)
    (d) A(1, 1, –1), B(2, 0, 2) and C(0, –2, 1)
    (e) A(3, 1, –4), B(2, –1, 2) and C(–3, 2, 1)
    (f) A(1, 0, 1), B(1, 1, 1) and C(2, 1, –1)
    (g) A(2, 1, 4), B(–1, 1, 0) and C(3, 0, 4)
    (h) O(0, 0, 0), P(1, –1, 2) and Q(3, 2, –1)
    (i) K(2, 1, –4), L(3, –2, 5) and M(–4, 1, 2)

13. (a) Find the equation of the plane containing the points A(5, 7, –1), B(2, –3, 6) and C(1, –4, 7).

    (b) Show that the point D(6, 1, 2) also lies in the plane in (a).
    [The points A, B, C and D are said to be *coplanar* since the points all lie in the same plane.]

14. Show that the points A(2, –1, 0), B(5, 7, 6), C(–3, 3, –3) and D(–1, –9, –6) are *coplanar* (that is, all lie in the same plane).

[*Hint*: Find the equation of the plane containing the points A, B and C, and then show that the point D also lies in this plane.]

15. Show that the points A(4, 2, 11), B(6, –9, 5), C(–3, 2, 8) and D(15, –20, 2) are *coplanar* (see question 14).

16. Calculate the size of the **acute** angle between the planes π_1 and π_2 in each case.

(a) $\pi_1: \; x + 2y + z = 5$
$\pi_2: \; x + y = 0$

(b) $\pi_1: \; 2x - 2y + 6z = 11$
$\pi_2: \; 5x + 9y + 13z = 5$

(c) $\pi_1: \; 2x + y - 2z = 5$
$\pi_2: \; 3x - 6y - 2z = 7$

(d) $\pi_1: \; x + y + 4z = -1$
$\pi_2: \; 2x - 3y + 4z = -5$

(e) $\pi_1: \; 2x - y = 0$
$\pi_2: \; x + y + z = 0$

(f) $\pi_1: \; 3x + 5y - 2z = 11$
$\pi_2: \; 4x - 2y - 3z = 15$

17. Show that the planes with equations $2x + 3y - 2z = 1$ and $4x - 2y + z = 1$ are perpendicular.

18. (a) The plane π_1 contains the vectors 2**i** + **j** and 3**i** + 2**k**.
Find a vector normal to the plane π_1.

(b) The plane π_2 contains the vectors **i** + 3**j** – **k** and **i** + **j** – **k**.
Find a vector normal to the plane π_2.

(c) Hence calculate the acute angle between the planes π_1 and π_2.

19. (a) Find the equation of the plane π_1 containing the points O(0, 0, 0), A(1, 0, 1) and B(0, 1, 1).

(b) Find the equation of the plane π_2 containing the points P(1, –1, 1), Q(3, 1, –2) and R(0, 2, –1).

(c) Calculate the acute angle between the planes π_1 and π_2.

20. O is the point (0, 0, 0), A is (1, 2, 3), B is (2, 3, 1) and C is (3, 0, –1).

(a) The plane π_1 contains the points A, B and C.
Find the equation of plane π_1.

(b) The plane π_2 contains the points O, B and C.
Find the equation of plane π_2.

(c) Calculate the acute angle between planes π_1 and π_2.

ANSWERS

1. (a) $2x+3y+z=12$ (b) $5x+4y-3z=17$ (c) $2x-3y+z=-1$
(d) $-4x+6y+7z=101$ (e) $x-3y+2z=-5$ (f) $x+2y-2z=-7$

2. (a) $x+y+3z=13$
(b) $2x-5y+2z=18$
(c) $-3x+3y-5z=-18$ (or $3x-3y+5z=18$)
(d) $8x-2y+z=-59$

3. (a) $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ (b) $2x-y+3z=7$

4. (a) $-4\mathbf{i}+2\mathbf{j}-8\mathbf{k}$ (b) $-4x+2y-8z=-2$ (or $2x-y+4z=1$)

5. (a) $-2\mathbf{i}-18\mathbf{j}-8\mathbf{k}$ (b) $-2x-18y-8z=-22$ (or $x+9y+4z=11$)

6. (a) $7\mathbf{i}-6\mathbf{j}-\mathbf{k}$ (b) $7x-6y-z=0$

7. (a) $6\mathbf{i}-5\mathbf{j}$ (b) $6x-5y=-27$

8. (a) $-15\mathbf{i}+7\mathbf{j}-9\mathbf{k}$ (b) $-15x+7y-9z=0$ (or $15x-7y+9z=0$)

9. (a) $-5\mathbf{i}+\mathbf{j}+2\mathbf{k}$ (b) $-5x+y+2z=-5$ (or $5x-y-2z=5$)

10. (a) $\overrightarrow{AB}=\begin{pmatrix}-2\\-2\\2\end{pmatrix}$, $\overrightarrow{AC}=\begin{pmatrix}-1\\3\\-2\end{pmatrix}$; $\boldsymbol{n}=-2\mathbf{i}-6\mathbf{j}-8\mathbf{k}$
(b) $-2x-6y-8z=-22$ (or $x+3y+4z=11$)

11. (a) $\overrightarrow{OA}=\begin{pmatrix}1\\2\\1\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}-2\\1\\2\end{pmatrix}$; $\boldsymbol{n}=3\mathbf{i}-4\mathbf{j}+5\mathbf{k}$
(b) $3x-4y+5z=0$

12. (a) $x+y=3$
(b) $10x+y-14z=-9$
(c) $-40x-24y+16z=-16$ (or $5x+3y-2z=2$)
(d) $7x-5y-4z=6$
(e) $-16x-31y-13z=-27$ (or $16x+31y+13z=27$)
(f) $-2x-z=-3$ (or $2x+z=3$)
(g) $-4x-4y+3z=0$ (or $4x+4y-3z=0$)
(h) $-3x+7y+5z=0$ (or $3x-7y-5z=0$)
(i) $-18x-60y-18z=-24$ (or $3x+10y+3z=4$)

13. (a) $-3x-4y-7z=-36$ (or $3x+4y+7z=36$)

16. (a) 30° (b) 50·5° (c) 79·0° (d) 49·0° (e) 75·0° (f) 76·1°

17. Show that the angle between the planes is 90°.

18. (a) $2\mathbf{i}-4\mathbf{j}-3\mathbf{k}$ (b) $-2\mathbf{i}-2\mathbf{k}$ (c) 82·5°

19. (a) $-x-y+z=0$ (or $x+y-z=0$) (b) $5x+7y+8z=6$ (c) 78·7°

20. (a) $-8x-4z=-20$ (or $2x+z=5$)
(b) $-3x+5y-9z=0$ (or $3x-5y+9z=0$)
(c) 51·3°