ADVANCED HIGHER MATHEMATICS

MATRICES

INTRODUCTION

A **matrix** is simply a **rectangular array of numbers.**

A matrix with m rows and n columns is said to have **order** $m \times n$.

The **entry** (or element) in row i and column j of the matrix A is denoted by a_{ij}.

Examples

(1) $A = \begin{pmatrix} 2 & 4 & -1 \\ 3 & 1 & 2 \end{pmatrix}$ is a 2×3 matrix.

Check that the entry $a_{13} = -1$ and that the entry $a_{21} = 3$.

(2) $B = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$ is a 2×2 matrix.

Check that the entry $b_{12} = 0$.

The matrix B is known as a **square matrix** since the order of matrix B is of the form $n \times n$.

(3) $C = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ is a 3×1 matrix.

Check that the entry $c_{31} = 2$.

The matrix C is knows as a **column** matrix since the order of matrix C is of the form $m \times 1$.

(4) $D = \begin{pmatrix} 1 & 3 & 5 & -1 \end{pmatrix}$ is a 1×4 matrix.

Check that the entry $d_{13} = 5$.

The matrix D is known as a **row** matrix since the order of matrix D is of the form $1 \times n$.

EQUAL MATRICES

Two matrices are **equal** if and only if the two matrices are of the same order and **all** pairs of corresponding entries are equal.

Worked Example

Given that $\begin{pmatrix} 2x & 0 \\ -1 & x+y \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ -1 & 1 \end{pmatrix}$, find the values of x and y.

Solution

Equating entries: $2x = 6 \Rightarrow x = 3$

$x + y = 1 \Rightarrow 3 + y = 1$

$\Rightarrow y = -2$

Hence $x = 3$ and $y = -2$.

[*Note that the matrices* $\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$ *and* $\begin{pmatrix} 2 & 1 & 0 \\ 3 & 5 & 0 \end{pmatrix}$ *are not equal, since the matrices have different order.*]

ADDITION AND SUBTRACTION OF MATRICES

Matrices of the same order can be added/subtracted by adding/subtracting corresponding entries.

Example 1

$$\begin{pmatrix} 3 & 4 & 5 \\ -2 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & -6 \end{pmatrix} = \begin{pmatrix} 3+2 & 4+(-1) & 5+2 \\ -2+3 & 1+1 & 3+(-6) \end{pmatrix}$$
$$= \begin{pmatrix} 5 & 3 & 7 \\ 1 & 2 & -3 \end{pmatrix}$$

Example 2

$$\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & -2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 5-1 & 1-(-2) \\ -2-2 & 3-4 \end{pmatrix}$$
$$= \begin{pmatrix} 4 & 3 \\ -4 & -1 \end{pmatrix}$$

SCALAR MULTIPLICATION

If k is a **scalar** (number), the matrix kA is formed by multiplying each entry of the matrix A by k.

Example 1

$$3\begin{pmatrix} 2 & 1 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 3\times 2 & 3\times 1 \\ 3\times(-3) & 3\times 5 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ -9 & 15 \end{pmatrix}$$

Example 2

$$-2\begin{pmatrix} 4 & 0 & -1 \\ -3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -2\times 4 & -2\times 0 & -2\times(-1) \\ -2\times(-3) & -2\times 2 & -2\times 1 \end{pmatrix}$$
$$= \begin{pmatrix} -8 & 0 & 2 \\ 6 & -4 & -2 \end{pmatrix}$$

Worked Example 3

Given the matrices $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} -2 & 4 \\ 5 & 0 \end{pmatrix}$, find the matrix $2A + 3B - 2C$.

Solution

$$2A + 3B - 2C = 2\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} + 3\begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} - 2\begin{pmatrix} -2 & 4 \\ 5 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 4 \\ 6 & -2 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ -3 & 6 \end{pmatrix} - \begin{pmatrix} -4 & 8 \\ 10 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 2+6-(-4) & 4+0-8 \\ 6+(-3)-10 & -2+6-0 \end{pmatrix}$$
$$= \begin{pmatrix} 12 & -4 \\ -7 & 4 \end{pmatrix}$$

[*Note that the distributive law applies for scalar multiplication.*
That is, $k(A+B) = kA + kB$ *for any matrices A and B of the same order, where k is a scalar.*]

THE TRANSPOSE OF A MATRIX

The **transpose** of a matrix A is denoted by A' and is formed by **swapping the rows and columns** of the matrix A.

The first **row** of the matrix A becomes the first **column** of the matrix A', and so on.

Example

If $A = \begin{pmatrix} 3 & 4 & 5 \\ 2 & -1 & 3 \end{pmatrix}$, then $A' = \begin{pmatrix} 3 & 2 \\ 4 & -1 \\ 5 & 3 \end{pmatrix}$.

Note that the matrix A is of order 2×3, whereas the matrix A' is of order 3×2.

This is true in general; if the matrix A is of order $m \times n$, then the matrix A' will be of order $n \times m$.

[*The transpose has the following properties:*

(1) $(A')' = A$ for any matrix A.

(2) $(A + B)' = A' + B'$ for any matrices A and B of the same order.

(3) $(kA)' = kA'$ for any matrix A and scalar k.]

SYMMETRIC AND SKEW-SYMMETRIC MATRICES

(1) A matrix A is said to be **symmetric** if $A' = A$.

Example

If $A = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 2 & -1 \\ 5 & -1 & 7 \end{pmatrix}$, then $A' = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 2 & -1 \\ 5 & -1 & 7 \end{pmatrix}$.

$A' = A$, so the matrix A is symmetric.

Note that a symmetric matrix is a square matrix which is symmetrical about the leading diagonal (the diagonal running from the top-left corner to the bottom-right corner of the matrix.)

(2) A matrix A is said to be **skew-symmetric** if $A' = -A$.

Example

If $A = \begin{pmatrix} 0 & 3 & -5 \\ -3 & 0 & 1 \\ 5 & -1 & 0 \end{pmatrix}$, then $A' = \begin{pmatrix} 0 & -3 & 5 \\ 3 & 0 & -1 \\ -5 & 1 & 0 \end{pmatrix}$.

$A' = -A$, so the matrix A is skew-symmetric.

Note that a skew-symmetric matrix must be a square matrix with all entries in the leading diagonal equal to zero.

YOU CAN NOW ATTEMPT THE WORKSHEET "MATRICES 1".

MATRIX MULTIPLICATION

The matrix product AB can only be formed if the number of columns of matrix A is the same as the number of rows of matrix B.

That is, the matrix product AB can only be formed if the order of matrix A is $m \times n$ and the order of matrix B is $n \times p$. The order of the matrix product AB is then $m \times p$.

The entry in row i and column j of the matrix AB is then found by multiplying row i of matrix A into column j of matrix B. This can be thought of as a "row into column" multiplication method.

[Note again the order of the matrices in the matrix product AB:

$$\begin{array}{ccc} A \quad B & & AB \\ (m \times n)(n \times p) & \rightarrow & m \times p \end{array}$$

It is convenient to think of the n's "cancelling out."]

Example 1

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$.

The matrix product AB can be formed since A is of order 2×2 and B is of order 2×2. The matrix AB will then be of order 2×2.

$$\begin{aligned} AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} &= \begin{pmatrix} 1\times 2 + 2\times 1 & 1\times 5 + 2\times 3 \\ 3\times 2 + 4\times 1 & 3\times 5 + 4\times 3 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 11 \\ 10 & 27 \end{pmatrix} \end{aligned}$$

The matrix product BA can also be formed since B is of order 2×2 and A is of order 2×2. The matrix BA will then be of order 2×2.

$$\begin{aligned} BA = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} &= \begin{pmatrix} 2\times 1 + 5\times 3 & 2\times 2 + 5\times 4 \\ 1\times 1 + 3\times 3 & 1\times 2 + 3\times 4 \end{pmatrix} \\ &= \begin{pmatrix} 17 & 24 \\ 10 & 14 \end{pmatrix} \end{aligned}$$

Note that the matrix products AB and BA are **not equal**. This is true in general for matrices A and B. If the matrix products AB and BA are equal, however, the matrices A and B are said to **commute**.
[The order in which matrices are multiplied is therefore crucial. In the matrix product AB, we say that A pre-multiplies B or that B post-multiplies A.]

Example 2

Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

The matrix product AB can be formed since A is of order 2×2 and B is of order 2×1. The matrix AB will then be of order 2×1.

$$AB = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \times 4 + (-1) \times (-2) \\ 3 \times 4 + 2 \times (-2) \end{pmatrix}$$
$$= \begin{pmatrix} 10 \\ 8 \end{pmatrix}$$

Note that the matrix product BA cannot be formed since B is of order 2×1 and A is of order 2×2.

Example 3

Let $P = \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & 4 & 5 \\ 6 & 1 & -2 \end{pmatrix}$.

The matrix product PQ can be formed since P is of order 2×2 and Q is of order 2×3. The matrix PQ will then be of order 2×3.

$$PQ = \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 & 5 \\ 6 & 1 & -2 \end{pmatrix}$$
$$= \begin{pmatrix} 4 \times 2 + 3 \times 6 & 4 \times 4 + 3 \times 1 & 4 \times 5 + 3 \times (-2) \\ -1 \times 2 + 2 \times 6 & -1 \times 4 + 2 \times 1 & -1 \times 5 + 2 \times (-2) \end{pmatrix}$$
$$= \begin{pmatrix} 26 & 19 & 14 \\ 10 & -2 & -9 \end{pmatrix}$$

Note that the matrix product QP cannot be formed since Q is of order 2×3 and P is of order 2×2.

Worked Example 4

Given the 2×2 matrix $M = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}$, find the matrices M^2, M^3 and M^4.

Solution

$$M^2 = MM = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 3\times 3 + (-2)\times 0 & 3\times(-2) + (-2)\times 3 \\ 0\times 3 + 3\times 0 & 0\times(-2) + 3\times 3 \end{pmatrix}$$
$$= \begin{pmatrix} 9 & -12 \\ 0 & 9 \end{pmatrix}$$

$$M^3 = MM^2 = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 9 & -12 \\ 0 & 9 \end{pmatrix}$$
$$= \begin{pmatrix} 3\times 9 + (-2)\times 0 & 3\times(-12) + (-2)\times 9 \\ 0\times 9 + 3\times 0 & 0\times(-12) + 3\times 9 \end{pmatrix}$$
$$= \begin{pmatrix} 27 & -54 \\ 0 & 27 \end{pmatrix}$$

[The matrix M^3 can also be found by forming the matrix product M^2M .]

$$M^4 = MM^3 = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 27 & -54 \\ 0 & 27 \end{pmatrix}$$
$$= \begin{pmatrix} 3\times 27 + (-2)\times 0 & 3\times(-54) + (-2)\times 27 \\ 0\times 27 + 3\times 0 & 0\times(-54) + 3\times 27 \end{pmatrix}$$
$$= \begin{pmatrix} 81 & -216 \\ 0 & 81 \end{pmatrix}$$

[The matrix M^4 can also be found by forming the matrix product M^3M or M^2M^2 .]

[If required, the matrix product ABC can be found by considering the matrix product $(AB)C$ or $A(BC)$. Note that the order $A \to B \to C$ must be preserved.]

YOU CAN NOW ATTEMPT THE WORKSHEET "MATRICES 2".

IDENTITY MATRICES

The matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the **2×2 identity matrix**.

Note that all the entries on the main diagonal are 1 and all other entries are zero in this identity matrix.

Now let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a general 2×2 matrix.

Then $IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1a+0c & 1b+0d \\ 0a+1c & 0b+1d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$.

Also, $AI = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1a+0b & 0a+1b \\ 1c+0d & 0c+1d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$.

Hence, for any 2×2 matrix A:

$IA = A$ and $AI = A$

In terms of matrix multiplication, the matrix I behaves like the number 1 in multiplication of real and complex numbers.

The 3×3 identity matrix is $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and behaves in the same way as the 2×2 identity matrix.

Worked Example 1

Given the 2×2 matrix $A = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$, find the values of the integers p and q such that $A^2 = pA + qI$.

Hence express the matrix A^3 in the form $xA + yI$, where x and y are integers.

Solution

$$A^2 = AA = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} 2\times 2 + (-1)\times 3 & 2\times(-1) + (-1)\times 5 \\ 3\times 2 + 5\times 3 & 3\times(-1) + 5\times 5 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -7 \\ 21 & 22 \end{pmatrix}$$

$$A^2 = pA + qI \Rightarrow \quad \begin{pmatrix} 1 & -7 \\ 21 & 22 \end{pmatrix} = p\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix} + q\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\Rightarrow \quad \begin{pmatrix} 1 & -7 \\ 21 & 22 \end{pmatrix} = \begin{pmatrix} 2p & -p \\ 3p & 5p \end{pmatrix} + \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}$$
$$\Rightarrow \quad \begin{pmatrix} 1 & -7 \\ 21 & 22 \end{pmatrix} = \begin{pmatrix} 2p+q & -p \\ 3p & 5p+q \end{pmatrix}$$

Equating entries: $\quad -p = -7 \quad \Rightarrow \quad p = 7$

$2p + q = 1 \quad \Rightarrow \quad 2(7) + q = 1$
$\Rightarrow \quad q = -13$

Hence $p = 7$ and $q = -13$.

Now $A^2 = pA + qI$, so $A^2 = 7A - 13I$.

$$A^3 = AA^2 = A(7A - 13I)$$
$$= 7A^2 - 13A \quad [\text{since } AI = A]$$
$$= 7(7A - 13I) - 13A$$
$$= 49A - 91I - 13A$$
$$= 36A - 91I$$

Hence $A^3 = 36A - 91I$.

[This technique can be extended repeatedly to find the matrices A^4, A^5, ... in the form $xA + yI$.]

Worked Example 2

Let $A = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$.

Show that $AB = kI$ for some real number k.
Hence obtain the matrix A^2B.

Solution

$$AB = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times1+2\times2+(-4)\times0 & 2\times(-1)+2\times3+(-4)\times1 & 2\times0+2\times4+(-4)\times2 \\ -4\times1+2\times2+(-4)\times0 & -4\times(-1)+2\times3+(-4)\times1 & -4\times0+2\times4+(-4)\times2 \\ 2\times1+(-1)\times2+5\times0 & 2\times(-1)+(-1)\times3+5\times1 & 2\times0+(-1)\times4+5\times2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$= 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 6I$$

Hence $AB = 6I$.

Now $A^2B = AAB = A(AB) = A(6I) = 6A$ [since $AI = I$]

Hence $A^2B = 6\begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 12 & 12 & -24 \\ -24 & 12 & -24 \\ 12 & -6 & 30 \end{pmatrix}$.

YOU CAN NOW ATTEMPT THE WORKSHEET "MATRICES 3."

DETERMINANTS OF SQUARE MATRICES

The **determinant** of a square matrix is a number associated with the matrix. Determinants have important applications and properties.

The determinant of a square matrix A is denoted by $\det A$ or $|A|$.

THE DETERMINANT OF A 2 × 2 MATRIX

The determinant of the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is found by calculating $ad - bc$.

$$\det A = ad - bc$$

This can also be written as:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 1

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 2 \times 4 - 1 \times 3 = 5$$

Example 2

$$\begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} = 3 \times 1 - 2 \times (-2) = 7$$

Worked Example 3

Find the two values of x such that $\begin{vmatrix} x & 2x \\ 2x & x+4 \end{vmatrix} = 1$.

Solution

$$\begin{vmatrix} x & 2x \\ 2x & x+4 \end{vmatrix} = x(x+4) - 2x \cdot 2x$$
$$= x^2 + 4x - 4x^2$$
$$= 4x - 3x^2$$

Hence $\quad 4x - 3x^2 = 1$

$\Rightarrow \quad 3x^2 - 4x + 1 = 0$

$\Rightarrow \quad (3x-1)(x-1) = 0$

$\Rightarrow \quad x = \frac{1}{3}$ or $x = 1$

YOU CAN NOW ATTEMPT QUESTIONS 1 TO 3 OF THE WORKSHEET "MATRICES 4".

THE DETERMINANT OF A 3×3 MATRIX

The determinant of a 3×3 matrix is found as follows:

STEP 1

Multiply the first entry of the top row by the determinant of the 2×2 matrix remaining when the row and column containing this entry are deleted.

STEP 2

Multiply the second entry of the top row by the determinant of the 2×2 matrix remaining when the row and column containing this entry are deleted.

STEP 3

Multiply the third entry of the top row by the determinant of the 2×2 matrix remaining when the row and column containing this entry are deleted.

STEP 4

Add the answers together in a $+ - +$ pattern.

Example 1

$$\begin{vmatrix} 1 & 2 & 1 \\ 4 & -2 & 3 \\ 2 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} - 2\begin{vmatrix} 4 & 3 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 4 & -2 \\ 2 & 1 \end{vmatrix}$$

$$\begin{aligned} &= 1\{-2 \times (-1) - 3 \times 1\} - 2\{4 \times (-1) - 3 \times 2\} + 1\{4 \times 1 - (-2) \times 2\} \\ &= 1(-1) - 2(-10) + 1(8) \\ &= 27 \end{aligned}$$

Example 2

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 4 & 5 \\ 0 & -1 & 2 \end{vmatrix} = 3\begin{vmatrix} 4 & 5 \\ -1 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 5 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 1 & 4 \\ 0 & -1 \end{vmatrix}$$

$$\begin{aligned} &= 3\{4 \times 2 - 5 \times (-1)\} + 2\{1 \times 2 - 5 \times 0\} + 0 \\ &= 3(13) + 2(2) \\ &= 43 \end{aligned}$$

Worked Example 3

Find the values of k for which the determinant of the 3×3 matrix

$$A = \begin{pmatrix} k & 1 & 1 \\ 2 & -2 & 3 \\ -1 & -1 & -k \end{pmatrix}$$

is zero.

Solution

$$\begin{aligned} \det A &= k\begin{vmatrix} -2 & 3 \\ -1 & -k \end{vmatrix} - 1\begin{vmatrix} 2 & 3 \\ -1 & -k \end{vmatrix} + 1\begin{vmatrix} 2 & -2 \\ -1 & -1 \end{vmatrix} \\ &= k\{(-2)(-k) - 3 \times (-1)\} - 1\{2(-k) - 3 \times (-1)\} + 1\{2 \times (-1) - (-2) \times (-1)\} \\ &= k(2k+3) - 1(-2k+3) + 1(-4) \\ &= 2k^2 + 3k + 2k - 3 - 4 \\ &= 2k^2 + 5k - 7 \end{aligned}$$

$$\begin{aligned} \det A = 0 \quad &\Rightarrow \quad 2k^2 + 5k - 7 = 0 \\ &\Rightarrow \quad (2k+7)(k-1) = 0 \\ &\Rightarrow \quad k = -\frac{7}{2} \text{ or } k = 1 \end{aligned}$$

INVERSE MATRICES

Consider the 2×2 matrices $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$.

$$AB = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 2\times 2 + 1\times(-3) & 2\times(-1)+1\times 2 \\ 3\times 2 + 2\times(-3) & 3\times(-1)+2\times 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= I$$

$$BA = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 2\times 2 + (-1)\times 3 & 2\times 1+(-1)\times 2 \\ -3\times 2 + 2\times 3 & -3\times 1+2\times 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= I$$

The **matrix** B is known as the inverse of the matrix A since $AB = I$ and $BA = I$.

The inverse of the matrix A is denoted by A^{-1}.

We write $A^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$.

It is important to realise that some square matrices do not have inverses. For example, given the 2×2 matrix $P = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$, it can be shown that no 2×2 matrix Q exists such that $PQ = I$. Hence the matrix P has no inverse.

It can be shown in general that if the determinant of a square matrix A is non-zero, the the inverse matrix A^{-1} exists and we say that the matrix A is **invertible** (or **non-singular**).

The inverse matrix A is such that

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I.$$

If the determinant of a square matrix A is zero, then an inverse matrix does not exist and we say that the matrix A is **non-invertible** (or **singular**).

Sometimes an inverse matrix can be found from a given matrix equation. If a matrix equation can be rewritten in the form $AB = I$, then $B = A^{-1}$.

Worked Example 1

Let $A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$.

(a) Show that the matrix A is invertible.

(b) Show that $A^2 = 5A - 2I$.

(c) Hence obtain the inverse matrix A^{-1}.

Solution

(a) $\det A = 2 \times 3 - 4 \times 1 = 2$

$\det A \neq 0$, so the matrix A is invertible.

(b)
$$A^2 = AA = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2\times 2 + 4\times 1 & 2\times 4 + 4\times 3 \\ 1\times 2 + 3\times 1 & 1\times 4 + 3\times 3 \end{pmatrix} = \begin{pmatrix} 8 & 20 \\ 5 & 13 \end{pmatrix}$$

$$5A - 2I = 5\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 20 \\ 5 & 15 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 20 \\ 5 & 13 \end{pmatrix}$$

Hence $A^2 = 5A - 2I$.

(c) $A^2 = 5A - 2I$

To find the inverse matrix A^{-1}, this matrix equation must be rewritten in the form $AB = I$ as follows:

$$A^2 = 5A - 2I$$
$$\Rightarrow \quad A^2 - 5A = -2I \quad [\times(-1)]$$
$$\Rightarrow \quad 5A - A^2 = 2I$$
$$\Rightarrow \quad A(5I - A) = 2I \quad [\text{since } AI = A]$$
$$\Rightarrow \quad A\left\{\frac{1}{2}(5I - A)\right\} = I$$

Hence $A^{-1} = \frac{1}{2}(5I - A)$.

Now $5I - A = 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -2 \\ -\frac{1}{2} & 1 \end{pmatrix}$$

[You can easily verify that $AA^{-1} = I$ and $A^{-1}A = I$.]

Worked Example 2

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix}$.

By considering the matrix product AB, obtain the inverse matrix A^{-1}.

Solution

$$AB = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times 1 + 1\times 4 + 1\times(-3) & 1\times 0 + 1\times(-2) + 1\times 2 & 1\times 1 + 1\times(-2) + 1\times 1 \\ 1\times 1 + 2\times 4 + 3\times(-3) & 1\times 0 + 2\times(-2) + 3\times 2 & 1\times 1 + 2\times(-2) + 3\times 1 \\ 1\times 1 + (-1)\times 4 + (-1)\times(-3) & 1\times 0 + (-1)\times(-2) + (-1)\times 2 & 1\times 1 + (-1)\times(-2) + (-1)\times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 2I$$

Now $\quad AB = 2I \qquad \Rightarrow \qquad A\left(\frac{1}{2}B\right) = I$

Hence $A^{-1} = \frac{1}{2}B = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 2 & -1 & -1 \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{pmatrix}$.

YOU CAN NOW ATTEMPT QUESTIONS 1 TO 12 OF THE WORKSHEET "MATRICES 5".

THE INVERSE OF A 2×2 MATRIX

It is possible to find the inverse of a 2×2 matrix or 3×3 matrix (if the inverse exists) from scratch.

Given the general 2×2 matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, it can be shown that the inverse matrix (if it exists) is given by

$$A^{-1}=\frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

i.e. $$A^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

It is straightforward to verify that $AA^{-1}=I$ and $A^{-1}A=I$.

Example 1

$$A=\begin{pmatrix} 2 & 3 \\ 7 & 11 \end{pmatrix}$$

$\det A=2\times11-3\times7=1$

$\det A\neq0$, so the matrix A is invertible.

$$A^{-1}=\frac{1}{1}\begin{pmatrix} 11 & -3 \\ -7 & 2 \end{pmatrix}=\begin{pmatrix} 11 & -3 \\ -7 & 2 \end{pmatrix}$$

[You can easily verify that $AA^{-1}=I$ and $A^{-1}A=I$.]

Example 2

$$P=\begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix}$$

$\det P=2\times3-(-4)\times(-1)=2$

$\det P\neq0$, so the matrix P is invertible.

$$P^{-1} = \frac{1}{2}\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 2 \\ \frac{1}{2} & 1 \end{pmatrix}.$$

[You can easily verify that $PP^{-1} = I$ and $P^{-1}P = I$.]

YOU CAN NOW ATTEMPT QUESTION 13 OF THE WORKSHEET "MATRICES 5".

THE INVERSE OF A 3×3 MATRIX

It can be shown that the same set of **elementary row operations** which change a square matrix A into the identity matrix I also change the identity matrix I into the inverse matrix A^{-1} (if it exists).

This fact can be used to find the inverse of a 3×3 matrix from scratch.

Worked Example

Assuming it exists, find the inverse of the 3×3 matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & -1 \\ 5 & 2 & 3 \end{pmatrix}$.

STEP 1

Form the augmented matrix $A \mid I$.

$$A \mid I = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -3 & -1 & 0 & 1 & 0 \\ 5 & 2 & 3 & 0 & 0 & 1 \end{array}\right)$$

The aim is to change the left-hand half of this matrix into the identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ using elementary row operations.

STEP 2

Use elementary row operations to change the first entries of row 2 and row 3 to zero:

$$R_2 \rightarrow R_2 - 2R_1$$
$$R_3 \rightarrow R_3 - 5R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -3 & -2 & -5 & 0 & 1 \end{array}\right)$$

STEP 3

Use an elementary row operation to change the second entry of row 3 into zero:

$$R_3 \rightarrow 5R_3 - 3R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -19 & -3 & 5 \end{array}\right)$$

STEP 4

Use an elementary row operation to change the third entry of row 2 to zero:

$$R_2 \rightarrow R_2 - 3R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -5 & 0 & 55 & 10 & -15 \\ 0 & 0 & -1 & -19 & -3 & 5 \end{array}\right)$$

STEP 5

Use an elementary row operation to change the second entry of row 1 to zero:

$$R_1 \rightarrow 5R_1 + R_2$$

$$\left(\begin{array}{ccc|ccc} 5 & 0 & 5 & 60 & 10 & -15 \\ 0 & -5 & 0 & 55 & 10 & -15 \\ 0 & 0 & -1 & -19 & -3 & 5 \end{array}\right)$$

STEP 6

Use an elementary row operation to change the third entry of row 1 to zero:

$$R_1 \rightarrow R_1 + 5R_3$$

$$\left(\begin{array}{ccc|ccc} 5 & 0 & 0 & -35 & -5 & 10 \\ 0 & -5 & 0 & 55 & 10 & -15 \\ 0 & 0 & -1 & -19 & -3 & 5 \end{array}\right)$$

STEP 7

Finally, use elementary row operations to change the first entry of row 1, the second entry of row 2 and the third entry of row 3 to 1.

$R_1 \rightarrow \frac{1}{5}R_1$

$R_2 \rightarrow -\frac{1}{5}R_2$

$R_3 \rightarrow -R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & -1 & 2 \\ 0 & 1 & 0 & -11 & -2 & 3 \\ 0 & 0 & 1 & 19 & 3 & -5 \end{array}\right)$$

This matrix is now of the form $I \mid A^{-1}$.

Hence $A^{-1} = \begin{pmatrix} -7 & -1 & 2 \\ -11 & -2 & 3 \\ 19 & 3 & -5 \end{pmatrix}$.

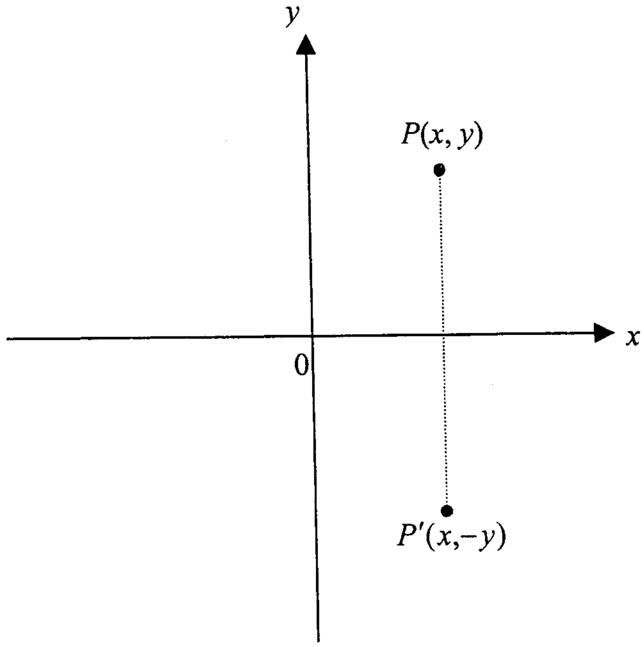
[You can verify that $AA^{-1} = I$ and $A^{-1}A = I$.]

[Note that this method can also be used to find the inverse of a 2×2 matrix, although it is usually much quicker to use the previous method.]

TRANSFORMATION MATRICES

REFLECTION IN THE x-AXIS

Under reflection in the x-axis, the image of the point $P(x, y)$ is $P'(x, -y)$, as shown in the diagram below.



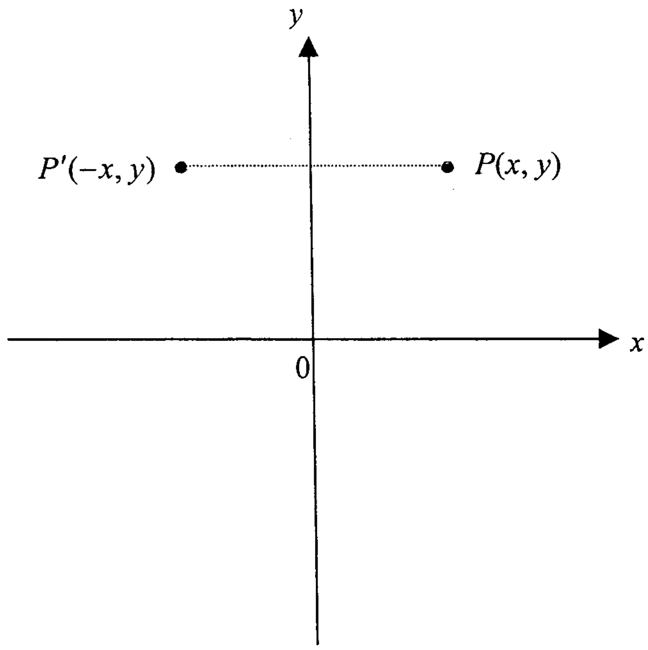
This can be represented by the matrix equation

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}.$$

We say that the 2×2 matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the matrix associated with reflection in the x-axis.

REFLECTION IN THE y-AXIS

Under reflection in the y-axis, the image of the point $P(x, y)$ is $P'(-x, y)$, as shown in the diagram below.



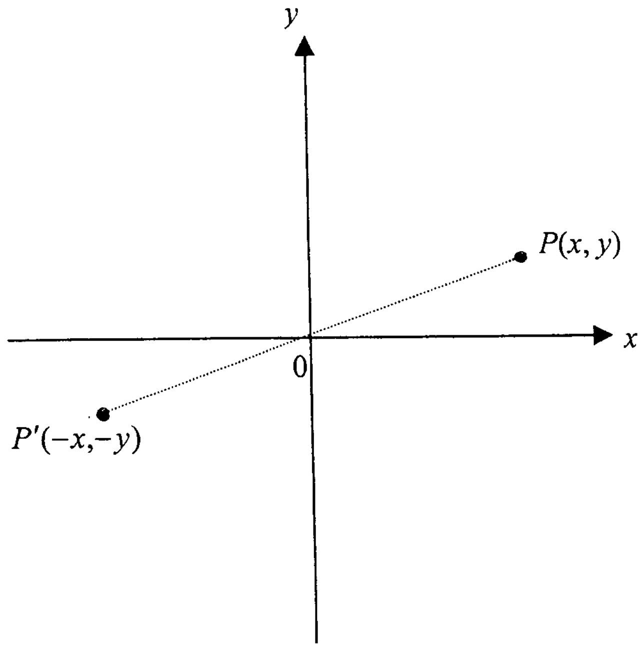
This can be represented by the matrix equation

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}.$$

We say that the 2×2 matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is the matrix associated with reflection in the y-axis.

REFLECTION IN THE ORIGIN

Under reflection in the origin, the image of the point $P(x, y)$ is $P'(-x, -y)$, as shown in the diagram below.



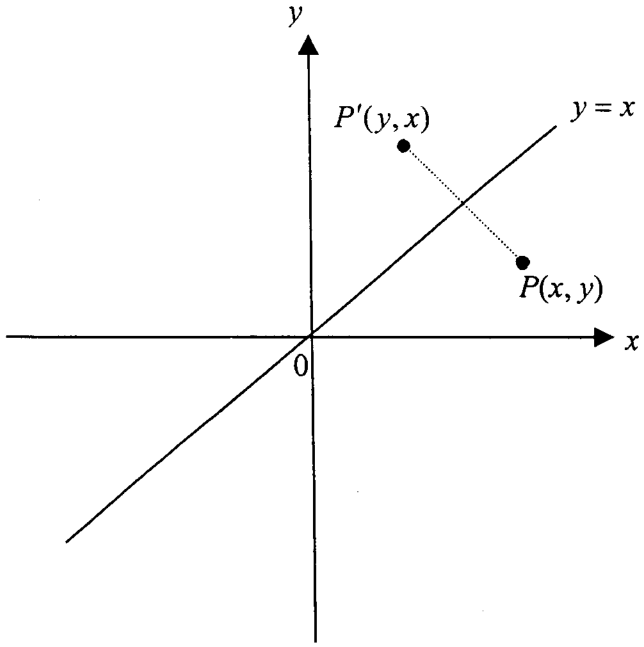
This can be represented by the matrix equation

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}.$$

We say that the 2×2 matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is the matrix associated with reflection in the origin.

REFLECTION IN THE LINE WITH EQUATION $y = x$

Under reflection in the line with equation $y = x$, the image of the point $P(x, y)$ is $P'(y, x)$, as shown in the diagram below.



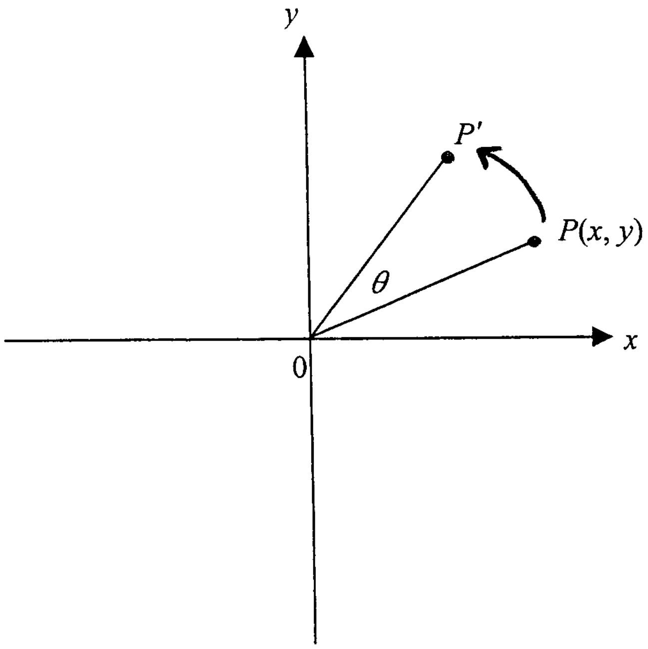
This can be represented by the matrix equation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}.$$

We say that the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the matrix associated with reflection in the line with equation $y = x$.

ROTATION ABOUT THE ORIGIN

Suppose that the point $P(x, y)$ is rotated through an angle of θ about the origin in an **anticlockwise** direction, as shown in the diagram below.



It can be shown that the 2×2 matrix associated with this rotation is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Worked Example 1

Write down the 2×2 matrix associated with an anticlockwise rotation of $\frac{\pi}{2}$ radians about the origin and hence find the image of the point $P(x, y)$ under this rotation.

Solution

The matrix associated with an anticlockwise rotation of $\frac{\pi}{2}$ radians about the origin is

$$\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$, hence the image of the point $P(x, y)$ is $P'(-y, x)$ under this rotation.

Worked Example 2

Write down the 2×2 matrix associated with a **clockwise** rotation of $30°$ about the origin.

Solution

Note that a **clockwise** rotation of $30°$ about the origin is equivalent to an **anticlockwise** rotation of $330°$ about the origin.

The matrix associated with an anticlockwise rotation of $330°$ about the origin is

$$\begin{pmatrix} \cos 330° & -\sin 330° \\ \sin 330° & \cos 330° \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Hence the 2×2 matrix associated with a **clockwise** rotation of $30°$ about the origin is $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$.

COMPOSITION OF TRANSFORMATIONS

Suppose we wish to reflect the point $P(x, y)$ in the x-axis and then reflect the image in the line with equation $y = x$.

Reflection in the x-axis

The matrix associated with reflection in the x-axis is $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The image of the point $P(x, y)$ under reflection in the x-axis is found by considering the matrix equation

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}.$$

Hence $P(x, y) \rightarrow P'(x,-y)$ under reflection in the x-axis.

Reflection in the line with equation $y = x$

The image point $P'(x,-y)$ is then reflected in the line with equation $y = x$.

Now the matrix associated with reflection in the line with equation $y = x$ is $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The image of the point $P'(x,-y)$ under reflection in the line with equation $y = x$ is found by considering the matrix equation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

Hence $P'(x,-y) \rightarrow P''(-y, x)$ under reflection in the line with equation $y = x$.

Hence $P(x, y) \rightarrow P''(-y, x)$ under the composite transformation.

Composite Transformation

Note that the overall effect of the composite transformation is represented by the matrix product

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ pre-multiplies $\begin{pmatrix} x \\ y \end{pmatrix}$ at the first stage of the composite transformation and the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then pre-multiples the answer $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ at the second stage.

The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the matrix associated with the composite transformation. This matrix is the matrix BA (note the order in which the matrices A and B are multiplied).

In general, if the matrix M_1 is the matrix associated with the transformation T_1 and the matrix M_2 is the matrix associated with the transformation T_2, then the matrix associated with the composite transformation T_1 followed by T_2 is M_2M_1.

Note the order in which the matrices M_1 and M_2 are multiplied - this order must be adhered to.

Worked Example

The point $P(x, y)$ is given an anticlockwise rotation of $\frac{\pi}{2}$ radians about the origin and the image is then reflected in the x-axis.

(a) Find the matrix associated with this composite transformation.

(b) Find the coordinates of the image of the point $P(x, y)$ under this composite transformation.

Solution

(a) The matrix associated with an anticlockwise rotation of $\frac{\pi}{2}$ radians about the origin is

$$M_1 = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The matrix associated with reflection in the x-axis is $M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The matrix associated with the composite transformation is M_2M_1.

$$M_2M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

(b) $$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$$

Hence the image of the point $P(x, y)$ under the composite transformation is $P'(-y, -x)$.

[Note that the overall effect of the composite transformation is equivalent to reflection in the line with equation $y = -x$.]