ADVANCED HIGHER MATHEMATICS

THREE DIMENSIONAL GEOMETRY

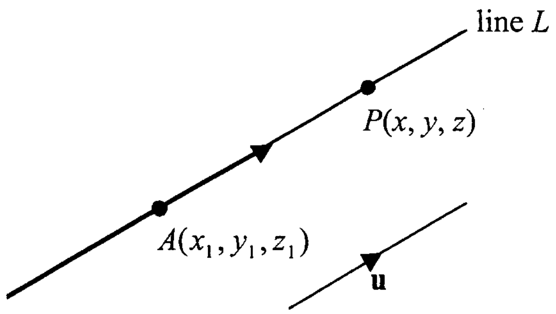
THE EQUATION OF A LINE IN THREE DIMENSIONS

Recall that the equation of a line in two dimensional space can be found when you know the coordinates of a point on the line and the gradient of the line. This information uniquely defines a line in two dimensional space.

The equation of a line in three dimensional space can be found when you know the coordinates of a point on the line and a **vector** in the direction of the line. This information uniquely defines a line in three dimensional space.

THE EQUATION OF A LINE IN SYMMETRIC FORM

Consider in general the line L which passes through the point $A(x_1, y_1, z_1)$ parallel to the vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ (the vector $\mathbf{u}$ is known as a **direction vector** for the line L). Let $P(x, y, z)$ be any point on the line L.



The vector $\overrightarrow{AP}$ is parallel to the vector $\mathbf{u}$.

Hence $\overrightarrow{AP} = t\mathbf{u}$ for some non-zero scalar t.

In component form, this can be written as $\begin{pmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{pmatrix} = t\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ta \\ tb \\ tc \end{pmatrix}$.

Equating components:

$$x - x_1 = ta \quad \Rightarrow \quad \frac{x - x_1}{a} = t$$

$$y - y_1 = tb \quad \Rightarrow \quad \frac{y - y_1}{b} = t$$

$$z - z_1 = tc \quad \Rightarrow \quad \frac{z - z_1}{c} = t$$

These equations can be concisely written in the form

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=t.$$

Thus the equation of the line can be written in the form

$$\boxed{\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}}$$

This is known as expressing the equation of the line L in **symmetric form.**

Note that the coordinates of a point on the line can be read from the numerators of the fractions and a vector parallel to the line can be read from the denominators of the fractions.

Worked Example 1

(a) The line L passes through the point $A(1, -2, 8)$ and is parallel to the vector $3\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}$. Find the equation of line L in symmetric form.

(b) Show that the point $B(-2, -7, -3)$ also lies on the line L.

Solution

(a) Equation of line L in symmetric form:

$$\frac{x-1}{3}=\frac{y+2}{5}=\frac{z-8}{11}$$

(b) At the point $B(-2, -7, -3)$: $\frac{x-1}{3}=\frac{-2-1}{3}=-1$

$$\frac{y+2}{5}=\frac{-7+2}{5}=-1$$

$$\frac{z-8}{11}=\frac{-3-8}{11}=-1$$

The coordinates of B satisfy $\frac{x-1}{3}=\frac{y+2}{5}=\frac{z-8}{11}$ and hence the point B lies on the line L.

Worked Example 2

Find, in symmetric form, the equation of the line passing through the points $A(2, -1, 3)$ and $B(3, 2, 5)$.

Solution

The vector $\overrightarrow{AB}$ is a vector in the direction of the line.

$$\overrightarrow{AB} = \begin{pmatrix} 3-2 \\ 2-(-1) \\ 5-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

Equation of line in symmetric form (using the coordinates of point A):

$$\frac{x-2}{1} = \frac{y+1}{3} = \frac{z-3}{2}$$

[Using the coordinates of point B, the equation of the line can also be expressed in the form

$$\frac{x-3}{1} = \frac{y-2}{3} = \frac{z-5}{2}.$$

It can be shown that this equation is in fact equivalent to the equation found using the coordinates of point A.]

[Note that the vector $\overrightarrow{BA}$ could also be used as a vector in the direction of the line.]

YOU CAN NOW ATTEMPT QUESTIONS 1 TO 3 OF THE WORKSHEET "EQUATIONS OF A STRAIGHT LINE IN 3-DIMENSIONS".

THE EQUATION OF A LINE IN PARAMETRIC FORM

Consider the equation of a line in the general symmetric form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = t.$$

Then:

$$\frac{x - x_1}{a} = t \quad \Rightarrow \quad x = x_1 + at$$

$$\frac{y - y_1}{b} = t \quad \Rightarrow \quad y = y_1 + bt$$

$$\frac{z - z_1}{c} = t \quad \Rightarrow \quad z = z_1 + ct$$

Expressing x, y and z in terms of the **parameter** t is known as expressing the equation of the line in **parametric form**.

Worked Example 1

A line passes through the point $A(3, -2, 5)$ and is parallel to the vector $2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$. Find the equation of this line in parametric form.

Solution

Equation of line in symmetric form:

$$\frac{x - 3}{2} = \frac{y + 2}{4} = \frac{z - 5}{-1} = t$$

Equation of line in parametric form:

$$x = 2t + 3, \quad y = 4t - 2, \quad z = -t + 5$$

Worked Example 2

Find, in parametric form, the equation of the line passing through the points $A(1, -2, -1)$ and $B(2, 3, 1)$.

Solution

The vector $\overrightarrow{AB}$ is a vector in the direction of the line.

$$\overrightarrow{AB} = \begin{pmatrix} 2-1 \\ 3-(-2) \\ 1-(-1) \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$$

Equation of line in symmetric form (using the coordinates of point A)

$$\frac{x-1}{1} = \frac{y+2}{5} = \frac{z+1}{2} = t$$

Equation of line in parametric form:

$$x = t+1, \quad y = 5t-2, \quad z = 2t-1$$

YOU CAN NOW ATTEMPT QUESTION 4 OF THE WORKSHEET "EQUATIONS OF A STRAIGHT LINE IN 3-DIMENSIONS".

We have already seen how the equation of a line given in symmetric form can easily be rewritten in parametric form. The following example illustrates how the equation of a line given in parametric form can be rewritten in symmetric form.

Worked Example 3

The equation of a line in parametric form is

$$x = 2t + 3, \quad y = t - 4, \quad z = -3t + 1.$$

Find the equation of this line in symmetric form.

Solution

$$x = 2t + 3 \quad \Rightarrow \quad \frac{x-3}{2} = t$$

$$y = t - 4 \quad \Rightarrow \quad \frac{y+4}{1} = t$$

$$z = -3t + 1 \quad \Rightarrow \quad \frac{z-1}{-3} = t$$

Hence the equation of the line in symmetric form is

$$\frac{x-3}{2} = \frac{y+4}{1} = \frac{z-1}{-3}$$

YOU CAN NOW ATTEMPT QUESTIONS 5 TO 9 OF THE WORKSHEET "EQUATIONS OF A STRAIGHT LINE IN 3-DIMENSIONS".

THE INTERSECTION OF TWO LINES

The point of intersection of two lines in three dimensional space can be found when the equations of both lines are expressed in parametric form.

Worked Example

Let L_1 and L_2 be the lines

$$L_1: \quad x = 8 - 2t, \quad y = -4 + 2t, \quad z = 3 + t$$

$$L_2: \quad \frac{x}{-2} = \frac{y+2}{-1} = \frac{z-9}{2}$$

Show that the lines L_1 and L_2 intersect and find their point of intersection.

Solution

The equation of line L_2 must firstly be expressed in parametric form:

$$L_2: \quad \frac{x}{-2} = \frac{y+2}{-1} = \frac{z-9}{2} = s$$

$$\Rightarrow \quad x = -2s, \quad y = -s - 2, \quad z = 2s + 9$$

[Note that these equations are expressed in terms of the parameter s since the equations of line L_1 are already expressed in terms of the parameter t.]

$$L_1: \quad x = 8 - 2t, \quad y = -4 + 2t, \quad z = 3 + t$$

$$L_2: \quad x = -2s, \quad y = -s - 2, \quad z = 2s + 9$$

Equating x-coordinates: $\quad 8 - 2t = -2s \quad \Rightarrow \quad 2t - 2s = 8 \quad$ ...(1)

Equating y-coordinates: $\quad -4 + 2t = -s - 2 \quad \Rightarrow \quad 2t + s = 2 \quad$ ...(2)

Equating z-coordinates: $\quad 3 + t = 2s + 9 \quad \Rightarrow \quad t - 2s = 6 \quad$ ...(3)

Solving equations (1) and (2) simultaneously gives $t = 2$ and $s = -2$.

These values of t and s **must** now be checked in equation (3):

$$t - 2s = 2 - 2 \times (-2) = 6$$

The values of t and s also satisfy equation (3), hence the lines L_1 and L_2 intersect.

[If the values of t and s did not also satisfy equation (3), then this would mean that the lines did not intersect (see later).]

The coordinates of the point of intersection can now be found by substitution:

$$x = 8 - 2t = 8 - 2 \times 2 = 4$$

$$y = -4 + 2t = -4 + 2 \times 2 = 0$$

$$z = 3 + t = 3 + 2 = 5$$

The point of intersection is therefore (4, 0, 5).

[*Alternatively*:

$$x = -2s = -2 \times (-2) = 4$$
$$y = -s - 2 = -(-2) - 2 = 0$$
$$z = 2s + 9 = 2 \times (-2) + 9 = 5$$]

[If two lines in three dimensional space do not intersect, the lines must either be **parallel** or **skew** (non-parallel but will never intersect).]

YOU CAN NOW ATTEMPT QUESTIONS 10 AND 11 OF THE WORKSHEET "EQUATIONS OF A STRAIGHT LINE IN 3-DIMENSIONS".

THE ANGLE BETWEEN TWO LINES

The angle between two intersecting lines L_1 and L_2 is clearly the angle between the direction vectors of the lines.

The angle between two intersecting lines L_1 and L_2 is therefore found as follows.

STEP 1

Find a vector **a** in the direction of the line L_1.

STEP 2

Find a vector **b** in the direction of line L_2.

STEP 3

The angle, θ, between the lines L_1 and L_2 is the angle between the direction vectors **a** and **b**.

Hence:

$$\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Note that the direction vector of a line can easily be read from the symmetric form of the equation of the line. The equation of each line should therefore be expressed in symmetric form before finding the angle between two lines.

Worked Example

Let L_1 and L_2 be the lines

$$L_1: \quad \frac{x+3}{-1} = \frac{y-4}{1} = \frac{z}{-1}$$

$$L_2: \quad x = 2t - 2, \quad y = -3t + 1, \quad z = t - 1$$

Given that the lines L_1 and L_2 intersect, calculate the size of the **acute** angle between the lines.

Solution

$$L_1: \quad \frac{x+3}{-1} = \frac{y-4}{1} = \frac{z}{-1}$$

The vector $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ is a vector in the direction of the line L_1.

The equation of line L_2 must be expressed in symmetric form:

$$L_2: \quad \frac{x+2}{2} = \frac{y-1}{-3} = \frac{z+1}{1} = t$$

The vector $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ is a vector in the direction of the line L_2.

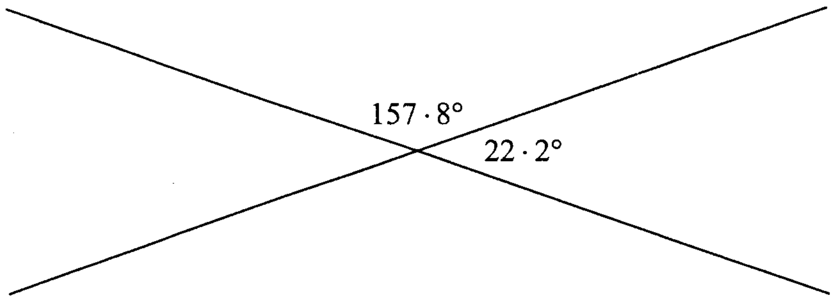
The angle, θ, between the lines L_1 and L_2 is the angle between the direction vectors **a** and **b**.

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \\
&= \frac{(-1)\times 2 + 1\times(-3) + (-1)\times 1}{\sqrt{(-1)^2 + 1^2 + (-1)^2}\sqrt{2^2 + (-3)^2 + 1^2}} \\
&= \frac{-6}{\sqrt{3}\sqrt{14}} \\
&= -0 \cdot 925...
\end{aligned}$$

$$\Rightarrow \qquad \theta = \cos^{-1}(-0 \cdot 925...) = 157 \cdot 8°$$

This is the **obtuse** angle between the two lines.

The **acute** angle between the two lines is $180° - 157 \cdot 8° = 22 \cdot 2°$.



YOU CAN NOW ATTEMPT QUESTIONS 12 TO 15 OF THE WORKSHEET "EQUATIONS OF A STRAIGHT LINE IN 3-DIMENSIONS".