SEQUENCES AND SERIES

SIGMA NOTATION

The notation $\sum_{k=a}^{b} f(k)$ is shorthand for the sum $f(a)+f(a+1)+f(a+2)+\ldots+f(b)$, where a and b are integers such that $a \leq b$.

Example 1

$$\begin{aligned}\sum_{k=1}^{3}(2k+1) &= (2\times 1+1)+(2\times 2+1)+(2\times 3+1)\\ &= 3+5+7\\ &= 15\end{aligned}$$

Example 2

$$\begin{aligned}\sum_{k=0}^{4}(-2)^k &= (-2)^0+(-2)^1+(-2)^2+(-2)^3+(-2)^4\\ &= 1+(-2)+4+(-8)+16\\ &= 11\end{aligned}$$

Example 3

$$\begin{aligned}\sum_{k=2}^{5}(3k^2-7) &= (3\times 2^2-7)+(3\times 3^2-7)+(3\times 4^2-7)+(3\times 5^2-7)\\ &= 5+20+41+68\\ &= 134\end{aligned}$$

Example 4

$$\sum_{k=-1}^{2}(2k+5)(k-3)$$

$$\begin{aligned}&= (2\times(-1)+5)(-1-3)+(2\times 0+5)(0-3)+(2\times 1+5)(1-3)+(2\times 2+5)(2-3)\\ &= (3)(-4)+(5)(-3)+(7)(-2)+(9)(-1)\\ &= -12-15-14-9\\ &= -50\end{aligned}$$

YOU CAN NOW ATTEMPT THE WORKSHEET "SEQUENCES AND SERIES 1".

USE OF PARTIAL FRACTIONS TO SUM SERIES

Partial fractions can be used to find the sum of certain series.

Worked Example

(a) Express $\frac{1}{4k^2-1}$ in partial fractions.

(b) Deduce that $\sum_{k=1}^{n} \frac{1}{4k^2-1} = \frac{1}{2} - \frac{1}{2(2n+1)}$.

(c) Evaluate $\sum_{k=1}^{\infty} \frac{1}{4k^2-1}$.

Solution

(a) Note that $\frac{1}{4k^2-1} = \frac{1}{(2k-1)(2k+1)}$ and the denominator contains distinct linear factors.

$$\frac{1}{(2k-1)(2k+1)} = \frac{A}{2k-1} + \frac{B}{2k+1}$$
$$= \frac{A(2k+1)+B(2k-1)}{(2k-1)(2k+1)}$$

$\Rightarrow \quad 1 = A(2k+1) + B(2k-1)$

Put $k = -\frac{1}{2}$ $\quad\Rightarrow\quad 1 = A(0) + B(-2)$
$\Rightarrow\quad -2B = 1$
$\Rightarrow\quad B = -\frac{1}{2}$

Put $k = \frac{1}{2}$ $\quad\Rightarrow\quad 1 = A(2) + B(0)$
$\Rightarrow\quad 2A = 1$
$\Rightarrow\quad A = \frac{1}{2}$

Hence $\frac{1}{4k^2-1} = \frac{\frac{1}{2}}{2k-1} - \frac{\frac{1}{2}}{2k+1}$
$$= \frac{1}{2(2k-1)} - \frac{1}{2(2k+1)}$$

(b) $$\sum_{k=1}^{n} \frac{1}{4k^2-1} = \sum_{k=1}^{n}\left(\frac{1}{2(2k-1)} - \frac{1}{2(2k+1)}\right)$$

$$= \frac{1}{2(1)} - \frac{1}{2(3)} \quad [k=1]$$

$$+ \frac{1}{2(3)} - \frac{1}{2(5)} \quad [k=2]$$

$$+ \frac{1}{2(5)} - \frac{1}{2(7)} \quad [k=3]$$

$$+ \cdots\cdots\cdots\cdots$$

$$+ \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)} \quad [k=n]$$

$$= \frac{1}{2} - \frac{1}{2(2n+1)}$$

[Note the cancellation of terms in pairs; the cancellation is most evident if the terms of the summation are written in a vertical fashion as shown.]

(c) Note that $$\sum_{k=1}^{\infty} \frac{1}{4k^2-1} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{4k^2-1}$$

$$= \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{2(2n+1)}\right).$$

Now as $n \to \infty$, $\frac{1}{2(2n+1)} \to 0$.

Hence $\sum_{k=1}^{\infty} \frac{1}{4k^2-1} = \frac{1}{2}$.

YOU CAN NOW ATTEMPT THE WORKSHEET "SEQUENCES AND SERIES 2".

THE METHOD OF DIFFERENCES

Sometimes an algebraic identity can be used to find the sum of a series.

Worked Example

(a) Show that $\frac{1}{3}k(k+1)(k+2) - \frac{1}{3}(k-1)k(k+1) = k(k+1)$.

(b) Deduce that $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$.

(c) Evaluate the sum of the series

$$(1\times 2) + (2\times 3) + (3\times 4) + \ldots + (99\times 100).$$

Solution

(a)
$$\frac{1}{3}k(k+1)(k+2) - \frac{1}{3}(k-1)k(k+1)$$
$$= \frac{1}{3}k(k+1)\{(k+2)-(k-1)\}$$
$$= \frac{1}{3}k(k+1)\{k+2-k+1\}$$
$$= \frac{1}{3}k(k+1)(3)$$
$$= k(k+1)$$

(b)
$$\sum_{k=1}^{n} k(k+1) = \sum_{k=1}^{n}\left\{\frac{1}{3}k(k+1)(k+2) - \frac{1}{3}(k-1)k(k+1)\right\}$$
$$= \frac{1}{3}(1)(2)(3) - \frac{1}{3}(0)(1)(2) \qquad [k=1]$$
$$+ \frac{1}{3}(2)(3)(4) - \frac{1}{3}(1)(2)(3) \qquad [k=2]$$
$$+ \frac{1}{3}(3)(4)(5) - \frac{1}{3}(2)(3)(4) \qquad [k=3]$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \frac{1}{3}n(n+1)(n+2) - \frac{1}{3}(n-1)n(n+1) \qquad [k=n]$$
$$= \frac{1}{3}n(n+1)(n+2) - 0$$
$$= \frac{n(n+1)(n+2)}{3}$$

[Note the cancellation of terms in pairs; the cancellation is most evident if the terms of the summation are written in a vertical fashion as shown.]

(c) $(1 \times 2) + (2 \times 3) + (3 \times 4) + \ldots + (99 \times 100)$

$= \sum_{k=1}^{99} k(k+1)$

$= \dfrac{(99)(99+1)(99+2)}{3}$ [using the result $\sum_{k=1}^{n} k(k+1) = \dfrac{n(n+1)(n+2)}{3}$]

$= \dfrac{99 \times 100 \times 101}{3}$

$= 333\ 300$

YOU CAN NOW ATTEMPT THE WORKSHEET "SEQUENCES AND SERIES 3".

USE OF STANDARD FORMULAE TO SUM SERIES

First note the following two properties of $\sum$:

PROPERTY 1

$$\sum_{k=1}^{n}\{f(k)+g(k)\}=\sum_{k=1}^{n}f(k)+\sum_{k=1}^{n}g(k)$$

PROOF

$$\sum_{k=1}^{n}\{f(k)+g(k)\}$$
$$=\{f(1)+g(1)\}+\{f(2)+g(2)\}+...+\{f(n)+g(n)\}$$
$$=\{f(1)+f(2)+...+f(n)\}+\{g(1)+g(2)+...+g(n)\}$$
$$=\sum_{k=1}^{n}f(k)+\sum_{k=1}^{n}g(k)$$

PROPERTY 2

If a is a constant, then $\sum_{k=1}^{n}af(k)=a\sum_{k=1}^{n}f(k)$.

PROOF

$$\sum_{k=1}^{n}af(k)=af(1)+af(2)+...+af(n)$$
$$=a\{f(1)+f(2)+...+f(n)\}$$
$$=a\sum_{k=1}^{n}f(k)$$

In particular, if a and b are constants:

$$\sum_{k=1}^{n}\{af(k)+bg(k)\}=\sum_{k=1}^{n}af(k)+\sum_{k=1}^{n}bg(k) \quad \text{[using property 1]}$$
$$=a\sum_{k=1}^{n}f(k)+b\sum_{k=1}^{n}g(k) \quad \text{[using property 2]}$$

This result extends to more than two functions.

[Note that this result also holds for integration:

$$\int\{af(x)+bg(x)\}dx=a\int f(x)dx+b\int g(x)dx$$

This is no coincidence, as integration is the limit of summation.]

The following standard formulae may be used to sum series:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

These formulae should be memorised. Note that $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$.

Proofs of these formulae are given in questions 2, 3 and 4 of the worksheet "SEQUENCES AND SERIES 3".

Worked Example 1

Evaluate $\sum_{k=1}^{10}(2k^3 - 3k)$.

Solution

$$\begin{aligned}\sum_{k=1}^{10}(2k^3 - 3k) &= 2\sum_{k=1}^{10} k^3 - 3\sum_{k=1}^{10} k \quad \text{[using the properties of } \textstyle\sum \text{]}\\ &= 2\left(\frac{10^2 \times 11^2}{4}\right) - 3\left(\frac{10 \times 11}{2}\right) \quad \text{[using the standard formula for } \textstyle\sum_{k=1}^{n} k^3 \text{ and } \textstyle\sum_{k=1}^{n} k \text{]}\\ &= 2(3025) - 3(55)\\ &= 5885\end{aligned}$$

Worked Example 2

Evaluate the sum of the series

$$(1\times 4^2)+(2\times 5^2)+(3\times 6^2)+\ldots+(15\times 18^2).$$

Solution

$$\begin{aligned}
&(1\times 4^2)+(2\times 5^2)+(3\times 6^2)+\ldots+(15\times 18^2)\\
&=\sum_{k=1}^{15} k(k+3)^2\\
&=\sum_{k=1}^{15} k(k^2+6k+9)\\
&=\sum_{k=1}^{15} (k^3+6k^2+9k)\\
&=\sum_{k=1}^{15} k^3+6\sum_{k=1}^{15} k^2+9\sum_{k=1}^{15} k\\
&=\left(\frac{15^2\times 16^2}{4}\right)+6\left(\frac{15\times 16\times 31}{6}\right)+9\left(\frac{15\times 16}{2}\right)\\
&=14400+6(1240)+9(120)\\
&=22\,920
\end{aligned}$$

Worked Example 3

Obtain an expression in terms of n for $\sum_{k=1}^{n}(2k^2-5)$.

Give your answer as a single algebraic fraction in its simplest form.

Solution

$$\begin{aligned}\sum_{k=1}^{n}(2k^2-5) &= 2\sum_{k=1}^{n}k^2-\sum_{k=1}^{n}5\\ &= 2\left\{\frac{n(n+1)(2n+1)}{6}\right\}-5n \qquad \left[\text{since } \sum_{k=1}^{n}5=5n\right]\\ &= \frac{n(n+1)(2n+1)}{3}-5n\\ &= \frac{n(n+1)(2n+1)}{3}-\frac{15n}{3}\\ &= \frac{n(n+1)(2n+1)-15n}{3}\\ &= \frac{n\{(n+1)(2n+1)-15\}}{3}\\ &= \frac{n\{2n^2+3n+1-15\}}{3}\\ &= \frac{n(2n^2+3n-14)}{3}\\ &= \frac{n(n-2)(2n+7)}{3}\end{aligned}$$

[Note that $\sum_{k=1}^{n}(2k^2-5)=2\sum_{k=1}^{n}k^2-5n$ and that $\sum_{k=1}^{n}(2k^2-5)\neq 2\sum_{k=1}^{n}k^2-5$.]

Worked Example 4

Evaluate $\sum_{k=25}^{45} k^2$.

Solution

$$\begin{aligned}\sum_{k=25}^{45} k^2 &= 25^2 + 26^2 + \ldots + 45^2 \\ &= (1^2 + 2^2 + \ldots + 45^2) - (1^2 + 2^2 + \ldots + 24^2) \\ &= \sum_{k=1}^{45} k^2 - \sum_{k=1}^{24} k^2 \\ &= \left(\frac{45 \times 46 \times 91}{6}\right) - \left(\frac{24 \times 25 \times 49}{6}\right) \\ &= 31395 - 4900 \\ &= 26\,495\end{aligned}$$

[With practice, $\sum_{k=25}^{45} k^2$ may be written as $\sum_{k=1}^{45} k^2 - \sum_{k=1}^{24} k^2$ directly.]

YOU CAN NOW ATTEMPT THE WORKSHEET "SEQUENCES AND SERIES 4".

Worked Example 5

(a) Obtain an expression for $\sum_{k=1}^{n} k(k+1)(2k-1)$ in terms of n.

Give your answer as a single algebraic fraction in its simplest form.

(b) Hence evaluate the sum of the series

$$(1\times 2\times 1)+(2\times 3\times 3)+(3\times 4\times 5)+(4\times 5\times 7)+\ldots+(20\times 21\times 39).$$

Solution

(a)

$$\sum_{k=1}^{n} k(k+1)(2k-1)$$

$$=\sum_{k=1}^{n} k(2k^2+k-1)$$

$$=\sum_{k=1}^{n} (2k^3+k^2-k)$$

$$=2\sum_{k=1}^{n} k^3+\sum_{k=1}^{n} k^2-\sum_{k=1}^{n} k$$

$$=2\left\{\frac{n^2(n+1)^2}{4}\right\}+\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}$$

$$=\frac{n^2(n+1)^2}{2}+\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}$$

$$=\frac{3n^2(n+1)^2}{6}+\frac{n(n+1)(2n+1)}{6}-\frac{3n(n+1)}{6}$$

$$=\frac{3n^2(n+1)^2+n(n+1)(2n+1)-3n(n+1)}{6}$$

$$=\frac{n(n+1)\{3n(n+1)+(2n+1)-3\}}{6}$$

$$=\frac{n(n+1)\{3n^2+3n+2n+1-3\}}{6}$$

$$=\frac{n(n+1)(3n^2+5n-2)}{6}$$

$$=\frac{n(n+1)(n+2)(3n-1)}{6}$$

(b) $(1\times 2\times 1)+(2\times 3\times 3)+(3\times 4\times 5)+(4\times 5\times 7)+\ldots+(20\times 21\times 39)$

$$=\sum_{k=1}^{20} k(k+1)(2k-1)$$

$$=\frac{20(20+1)(20+2)(3\times 20-1)}{6}$$ [using the result $\sum_{k=1}^{n} k(k+1)(2k-1)=\frac{n(n+1)(n+2)(3n-1)}{6}$]

$$=\frac{20(21)(22)(59)}{6}$$

$=90\ 860$

YOU CAN NOW ATTEMPT THE WORKSHEET "SEQUENCES AND SERIES 5".

ARITHMETIC SEQUENCES

Consider the sequence 1, 3, 5, 7, 9, ...

The first term is 1 and the terms increase by 2 each time.
This is an **arithmetic sequence** with first term $a = 1$ and **common difference** $d = 2$.

THE GENERAL ARITHMETIC SEQUENCE

The general arithmetic sequence has first term a and common difference d.
The n^{th} term is denoted by u_n.

$u_1 = a$

$u_2 = u_1 + d = a + d$

$u_3 = u_2 + d = (a + d) + d = a + 2d$

$u_4 = u_3 + d = (a + 2d) + d = a + 3d$

and so on.

In general:

$$u_n = a + (n - 1)d$$

This formula can be used to find particular terms in arithmetic sequences.

NOTE

The general arithmetic sequence is defined by the **recurrence relation** $u_{n+1} = u_n + d$, with first term $u_1 = a$.

Worked Example 1

Find the 25th term of the arithmetic sequence 2, 6, 10, 14, ...

Solution

$a = 2$ and $d = 4$

$$u_n = a + (n - 1)d \quad \Rightarrow \quad \begin{aligned} u_{25} &= a + 24d \\ &= 2 + 24 \times 4 \\ &= 98 \end{aligned}$$

Worked Example 2

Find a formula for the n^{th} term, u_n, of the arithmetic sequence 1, 4, 7, 10, ...

Solution

$a = 1$ and $d = 3$

$$u_n = a + (n-1)d \quad \Rightarrow \quad u_n = 1 + (n-1)3$$
$$= 1 + 3n - 3$$
$$= 3n - 2$$

Worked Example 3

The 3rd term of an arithmetic sequence is 40 and the 5th term is 30.

(a) Find the first term and the common difference.
(b) Find the 15th term of the sequence.

Solution

(a) $u_n = a + (n-1)d$

$u_3 = a + 2d = 40$ ...(1)
$u_5 = a + 4d = 30$ ...(2)

(2) − (1) $\Rightarrow$ $2d = -10$
$\Rightarrow$ $d = -5$

Sub. in (1): $a + 2d = 40$ $\Rightarrow$ $a - 10 = 40$
$\Rightarrow$ $a = 50$

The first term is 50 and the common difference is -5.

[Note that $d < 0$ means the terms in the sequence are decreasing.]

(b) $$u_n = a + (n-1)d \quad \Rightarrow \quad u_{15} = a + 14d$$
$$= 50 + 14 \times (-5)$$
$$= -20$$

[The 15th term can also be found using the fact that $u_{15} = u_5 + 10d$ or $u_{15} = u_3 + 12d$.]

Worked Example 4

In the arithmetic sequence beginning 2, 8, 14, 20, ..., which term is the first term to exceed 100?

Solution

$a = 2$ and $d = 6$

$u_n = a + (n-1)d \quad \Rightarrow \quad u_n = 2 + (n-1)6$

$$= 2 + 6n - 6$$
$$= 6n - 4$$

Set $u_n = 100 \quad \Rightarrow \quad 6n - 4 = 100$

$$\Rightarrow \quad 6n = 104$$
$$\Rightarrow \quad n = 17\frac{1}{3}$$

This means that the 18th term is the first term to exceed 100.

[*Check*: $u_{17} = 6 \times 17 - 4 = 98 < 100$

$u_{18} = 6 \times 18 - 4 = 104 > 100$]

[or, more formally, set $u_n > 100 \quad \Rightarrow \quad 6n - 4 > 100$

$$\Rightarrow \quad 6n > 104$$
$$\Rightarrow \quad n > 17\frac{1}{3},$$

which means that u_{18} is the first term to exceed 100.]

YOU CAN NOW ATTEMPT THE WORKSHEET "SEQUENCES AND SERIES 6".

ARITHMETIC SERIES

Consider the sum of the arithmetic series $1+2+3+4+...+100$ (100 terms).

Let $S = 1 + 2 + 3 + 4 + ... + 100$

Then $S = 100 + 99 + 98 + 97 + ... + 1$ [reversing the order of the terms]

Add: $2S = 101 + 101 + 101 + 101 + ... + 101$
$= 100 \times 101$

$$\Rightarrow \quad S = \frac{100 \times 101}{2} = 5050$$

A similar method can be used to find the sum of the first n terms of the general arithmetic sequence with first term a and common difference d.

Let S_n denote the sum of the first n terms of the general arithmetic sequence with first term a and common difference d.

Then $S_n = u_1 + u_2 + u_3 + ... + u_n$.

Recall that:

$u_1 = a$
$u_2 = a + d$
$u_3 = a + 2d$
....
$u_n = a + (n-1)d$

So $S_n = a + (a+d) + (a+2d) + ... + \{a+(n-1)d\}$

Then $S_n = \{a+(n-1)d\} + \{a+(n-2)d\} + \{a+(n-3)d\} + ... + a$

Add: $2S_n = \{2a+(n-1)d\} + \{2a+(n-1)d\} + \{2a+(n-1)d\} + ... + \{2a+(n-1)d\}$
$= n\{2a+(n-1)d\}$

$$\Rightarrow \quad S_n = \frac{n}{2}\{2a+(n-1)d\}$$

$$\boxed{S_n = \frac{n}{2}\{2a+(n-1)d\}}$$

This formula is used to find the sum of the first n terms of an arithmetic sequence.

Worked Example 1

Find the sum of the first 15 terms of the arithmetic series $3+8+13+18+\ldots$

Solution

$a=3$ and $d=5$

$$S_n=\frac{n}{2}\{2a+(n-1)d\} \quad \Rightarrow \quad S_{15}=\frac{15}{2}\{2(3)+14(5)\}$$
$$=7\cdot 5\times 76$$
$$=570$$

The sum of the first 15 terms is 570.

Worked Example 2

Find the sum of the arithmetic series $12+19+26+33+\ldots+285$.

Solution

$a=12$ and $d=7$

We must determine how many terms are in the series.

We know that $u_n=285$

$$\Rightarrow \quad a+(n-1)d=285$$
$$\Rightarrow \quad 12+(n-1)7=285$$
$$\Rightarrow \quad 12+7n-7=285$$
$$\Rightarrow \quad 7n+5=285$$
$$\Rightarrow \quad 7n=280$$
$$\Rightarrow \quad n=40$$

There are 40 terms in the series.

$$S_n=\frac{n}{2}\{2a+(n-1)d\} \quad \Rightarrow \quad S_{40}=\frac{40}{2}\{2(12)+39(7)\}$$
$$=20\times 297$$
$$=5940$$

The sum of the series is 5940.

Worked Example 3

Let u_n denote the n^{th} term of the arithmetic sequence 2, 10, 18, 26, ...

Let $S_n = \sum_{k=1}^{n} u_k$.

(a) Find a formula for S_n in terms of n.

(b) Find the least value of n for which $S_n > 1000$.

(c) Evaluate $\sum_{k=20}^{40} u_k$.

Solution

(a) $a = 2$ and $d = 8$

$$S_n = \frac{n}{2}\{2a + (n-1)d\} \quad \Rightarrow \quad S_n = \frac{n}{2}\{2(2) + (n-1)8\}$$

$$= \frac{n}{2}\{4 + 8n - 8\}$$

$$= \frac{n}{2}(8n - 4)$$

$$= 4n^2 - 2n$$

[*Check*: We know that $S_3 = 2 + 10 + 18 = 30$.
The formula gives $S_3 = 4 \times 3^2 - 2 \times 3 = 30$ √]

(b) Set $S_n = 1000 \Rightarrow \quad 4n^2 - 2n = 1000$

$$\Rightarrow \quad 4n^2 - 2n - 1000 = 0 \quad [\div 2]$$

$$\Rightarrow \quad 2n^2 - n - 500 = 0$$

This quadratic equation must be solved using the quadratic formula with $a = 2$, $b = -1$ and $c = -500$:

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{(-1)^2 - 4 \times 2 \times (-500)}}{2 \times 2}$$

$$= \frac{1 \pm \sqrt{4001}}{4}$$

$$= 16 \cdot 06\ldots \quad \text{or} \quad -15 \cdot 56\ldots$$

But $n > 0$, so $n = 16 \cdot 06\ldots$

This means that at least 17 terms must be added to give a sum exceeding 1000.
Hence the least value of n for which $S_n > 1000$ is $n = 17$.

[*Check*: $S_n = 4n^2 - 2n$

$S_{16} = 4 \times 16^2 - 2 \times 16 = 992 < 1000$
$S_{17} = 4 \times 17^2 - 2 \times 17 = 1122 > 1000$]

(c) $$\sum_{k=20}^{40} u_k = u_{20} + u_{21} + u_{22} + \ldots + u_{40}$$
$$= (u_1 + u_2 + \ldots + u_{40}) - (u_1 + u_2 + \ldots + u_{19})$$
$$= S_{40} - S_{19}$$

$S_{40} = 4 \times 40^2 - 2 \times 40 = 6320$
$S_{19} = 4 \times 19^2 - 2 \times 19 = 1406$

Hence $\sum_{k=20}^{40} u_k = 6320 - 1406 = 4914$.

Worked Example 4

The terms of a sequence are given by $u_k = 11 - 2k$, $k \geq 1$.

(a) Obtain a formula for S_n, where $S_n = \sum_{k=1}^{n} u_k$.

(b) Find the values of n for which $S_n = 21$.

Solution

(a) $u_k = 11 - 2k$

$u_1 = 11 - 2 \times 1 = 9$
$u_2 = 11 - 2 \times 2 = 7$
$u_3 = 11 - 2 \times 3 = 5$
$u_4 = 11 - 2 \times 4 = 3$

Hence the sequence is actually an arithmetic sequence with first term $a = 9$ and common difference $d = -2$.

$$S_n = \frac{n}{2}\{2a + (n-1)d\} \quad \Rightarrow \quad S_n = \frac{n}{2}\{2(9) + (n-1)(-2)\}$$
$$= \frac{n}{2}\{18 - 2n + 2\}$$
$$= \frac{n}{2}(20 - 2n)$$
$$= 10n - n^2$$

[*Check*: We know that $S_3 = 9 + 7 + 5 = 21$.
The formula gives $S_3 = 10 \times 3 - 3^2 = 21$ √]

(b) $S_n = 21 \Rightarrow 10n - n^2 = 21$
$\Rightarrow 10n - n^2 - 21 = 0 \quad [\times(-1)]$
$\Rightarrow n^2 - 10n + 21 = 0$
$\Rightarrow (n-3)(n-7) = 0$
$\Rightarrow n = 3$ or $n = 7$

Worked Example 5

The sum, S_n, of the first n terms of a sequence u_1, u_2, u_3, ... is given by $S_n = 8n - n^2$, $n \geq 1$.

(a) Calculate the values of u_1, u_2 and u_3 and state what type of sequence this is.

(b) Obtain a simple formula for u_n in terms of n.

Solution

(a) $S_n = 8n - n^2$

$$S_1 = 8 \times 1 - 1^2 = 7$$
$$S_2 = 8 \times 2 - 2^2 = 12$$
$$S_1 = 8 \times 3 - 3^2 = 15$$

$$u_1 = S_1 = 7$$
$$u_2 = S_2 - S_1 = 12 - 7 = 5$$
$$u_3 = S_3 - S_2 = 15 - 12 = 3$$

The sequence is an arithmetic sequence with first term $a = 7$ and common difference $d = -2$.

(b) $u_n = a + (n-1)d \quad \Rightarrow \quad u_n = 7 + (n-1)(-2)$

$$= 7 - 2n + 2$$
$$= 9 - 2n$$

YOU CAN NOW ATTEMPT THE WORKSHEET "SEQUENCES AND SERIES 7".

GEOMETRIC SEQUENCES

Consider the sequence 1, 2, 4, 8, 16, ...

The first term is 1 and the terms are multiplied by 2 each time.
This is a **geometric sequence** with first term $a = 1$ and **common ratio** $r = 2$.

THE GENERAL GEOMETRIC SEQUENCE

The general geometric sequence has first term a and common ratio r.
The n^{th} term is denoted by u_n.

$u_1 = a$

$u_2 = u_1 \times r = ar$

$u_3 = u_2 \times r = ar \times r = ar^2$

$u_4 = u_3 \times r = ar^2 \times r = ar^3$

and so on.

In general:

$$u_n = ar^{n-1}$$

This formula can be used to find particular terms in geometric sequences.

NOTE

The general geometric sequence is defined by the **recurrence relation** $u_{n+1} = ru_n$, with first term $u_1 = a$.

Worked Example 1

Find the 10^{th} term of the geometric sequence 3, 12, 48, 192, ...

Solution

$a = 3$ and $r = 4$

$u_n = ar^{n-1} \quad \Rightarrow \quad u_{10} = ar^9$

$= 3 \times 4^9$

$= 786\,432$

Worked Example 2

A geometric sequence of **positive** terms has third term 18 and seventh term 1458.
Find the fifth term of this sequence.

Solution

$u_n = ar^{n-1}$

$u_3 = ar^2 = 18 \qquad ...(1)$

$u_7 = ar^6 = 1458 \qquad ...(2)$

$(2) \div (1) \quad \Rightarrow \quad \dfrac{ar^6}{ar^2} = \dfrac{1458}{18}$

$\Rightarrow \quad r^4 = 81$

$\Rightarrow \quad r = \pm 3$

But all terms are positive, so $r \neq -3$.
Hence $r = 3$.

Sub. in (1): $ar^2 = 18 \quad \Rightarrow \quad a \times 3^2 = 18$

$\Rightarrow \quad 9a = 18$

$\Rightarrow \quad a = 2$

$u_n = ar^{n-1} \quad \Rightarrow \quad u_5 = ar^4$

$= 2 \times 3^4$

$= 162$

[The 5th term can also be found using the fact that $u_5 = u_3 \times r^2$ or $u_5 = \dfrac{u_7}{r^2}$.]

YOU CAN NOW ATTEMPT THE WORKSHEET "SEQUENCES AND SERIES 8".

GEOMETRIC SERIES

Let S_n be the sum of the first n terms of the geometric sequence with first term a and common ratio r.

Then $S_n = u_1 + u_2 + u_3 + ... + u_n$.

Recall that:

$$u_1 = a$$
$$u_2 = ar$$
$$u_3 = ar^2$$
$$\dots\dots\dots$$
$$u_n = ar^{n-1}$$

So $$S_n = a + ar + ar^2 + ... + ar^{n-1}$$

Then $$rS_n = \quad ar + ar^2 + ... + ar^{n-1} + ar^n$$

Subtract: $$S_n - rS_n = a - ar^n$$

$$\Rightarrow \quad S_n(1-r) = a(1-r^n)$$

$$\Rightarrow \quad S_n = \frac{a(1-r^n)}{1-r}, \quad r \neq 1$$

An alternative formula for S_n is $S_n = \dfrac{a(r^n - 1)}{r-1}$, $r \neq 1$.

The two formulae above for S_n are equivalent and either formula can be used to find the sum of the first n terms of a geometric sequence. However, for arithmetic simplicity, it is recommended that these formulae are used as follows:

- Use the formula $S_n = \dfrac{a(r^n - 1)}{r-1}$ when $r > 1$.
- Use the formula $S_n = \dfrac{a(1-r^n)}{1-r}$ when $r < 1$.

$$S_n = \frac{a(r^n - 1)}{r-1}, \quad r > 1$$

$$S_n = \frac{a(1-r^n)}{1-r}, \quad r < 1$$

NOTE

When $r = 1$: $u_1 = a$, $u_2 = a$, $u_3 = a$, ..., $u_n = a$ and $S_n = na$.

Worked Example 1

Find the sum of the first 9 terms of the geometric series $4 + 8 + 16 + 32 + \ldots$

Solution

$a = 4$ and $r = 2$

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \Rightarrow \quad S_9 = \frac{4(2^9 - 1)}{2 - 1}$$

$$= \frac{4(2^9 - 1)}{1}$$

$$= 2044$$

Worked Example 2

Evaluate $\sum_{k=1}^{20} (0 \cdot 9)^k$, giving your answer correct to 3 decimal places.

Solution

$$\sum_{k=1}^{20} (0 \cdot 9)^k = 0 \cdot 9 + (0 \cdot 9)^2 + (0 \cdot 9)^3 + \ldots + (0 \cdot 9)^{20}$$

This is a geometric series of 20 terms with $a = 0 \cdot 9$ and $r = 0 \cdot 9$.

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad \Rightarrow \quad S_{20} = \frac{0 \cdot 9\{1 - (0 \cdot 9)^{20}\}}{1 - 0 \cdot 9}$$

$$= \frac{0 \cdot 9\{1 - (0 \cdot 9)^{20}\}}{0 \cdot 1}$$

$$= 7 \cdot 906 \quad \text{(3 dp)}$$

Hence $\sum_{k=1}^{20} (0 \cdot 9)^k = 7 \cdot 906$ (3 dp).

Worked Example 3

Evaluate the sum of the geometric series $4 + 20 + 100 + \ldots + 62\,500$.

Solution

$a = 4$ and $r = 5$

We must determine how many terms are in the series.

We know that $u_n = 62500 \quad \Rightarrow \quad ar^{n-1} = 62500$

$$\Rightarrow \quad 4 \times 5^{n-1} = 62500$$
$$\Rightarrow \quad 5^{n-1} = 15625$$

This equation can be solved by taking natural logarithms of both sides of the equation:

$$\ln(5^{n-1}) = \ln 15625$$
$$\Rightarrow \quad (n-1)\ln 5 = \ln 15625$$
$$\Rightarrow \quad n - 1 = \frac{\ln 15625}{\ln 5}$$
$$\Rightarrow \quad n - 1 = 6$$
$$\Rightarrow \quad n = 7$$

There are 7 terms in the series.

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \Rightarrow \quad S_7 = \frac{4(5^7 - 1)}{5 - 1}$$
$$= \frac{4(5^7 - 1)}{4}$$
$$= 78\,124$$

The sum of the series is 78 124.

[The equation $5^{n-1} = 15625$ can also be solved as follows. By evaluating powers of 5, we find that $5^6 = 15625$. Hence, by comparison, $n - 1 = 6 \quad \Rightarrow \quad n = 7$]

Worked Example 4

Find the least number of terms of the geometric series $4 + 12 + 36 + 108 + \ldots$ which must be added to give a sum exceeding 1 000 000.

Solution

$a = 4$ and $r = 3$

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \Rightarrow \quad S_n = \frac{4(3^n - 1)}{3 - 1}$$

$$= \frac{4(3^n - 1)}{2}$$

$$= 2(3^n - 1)$$

Set $S_n = 1\ 000\ 000$ $\Rightarrow$ $2(3^n - 1) = 1\ 000\ 000$

$\Rightarrow$ $3^n - 1 = 500\ 000$

$\Rightarrow$ $3^n = 500\ 001$

$\Rightarrow$ $\ln(3^n) = \ln 500001$

$\Rightarrow$ $n \ln 3 = \ln 500001$

$\Rightarrow$ $n = \dfrac{\ln 500001}{\ln 3} = 11 \cdot 94\ldots$

This means that at least 12 terms must be added to give a sum exceeding 1 000 000.

[*Check*: $S_n = 2(3^n - 1)$

$S_{11} = 2(3^{11} - 1) = 354\ 292 < 1\ 000\ 000$

$S_{12} = 2(3^{12} - 1) = 1\ 062\ 880 > 1\ 000\ 000$]

Worked Example 5

A line 315 centimetres in length is divided into six parts such that the lengths of the parts form a geometric sequence.

Given that the length of the longest part is 32 times the length of the shortest part, find the length of the shortest part.

Solution

Let u_1 be the length of the shortest part, u_2 be the length of the second-shortest part, ..., u_6 be the length of the longest part.

Then $u_1, u_2, ..., u_6$ is a geometric sequence.

The length of the longest part is 32 times the length of the shortest part, so $u_6 = 32u_1$.

Now $u_n = ar^{n-1}$ for a geometric sequence, so $u_6 = ar^5$ and $u_1 = a$.

$$u_6 = 32u_1 \quad \Rightarrow \quad ar^5 = 32a$$
$$\Rightarrow \quad r^5 = 32$$
$$\Rightarrow \quad r = 2$$

The length of the line is 315 cm, so $u_1 + u_2 + ... + u_6 = 315$; that is, $S_6 = 315$.

$$\text{Now } S_n = \frac{a(r^n - 1)}{r - 1} \quad \Rightarrow \quad S_6 = \frac{a(2^6 - 1)}{2 - 1} = \frac{a(63)}{1} = 63a$$

$$S_6 = 315 \quad \Rightarrow \quad 63a = 315$$
$$\Rightarrow \quad a = 5$$

The length of the shortest part is 5 centimetres.

YOU CAN NOW ATTEMPT THE WORKSHEET "SEQUENCES AND SERIES 9".

INFINITE GEOMETRIC SERIES

Consider the sum of the first n terms of the geometric series $1 + 2 + 4 + 8 + ...$

$a = 1$ and $r = 2$

$$S_n = \frac{a(r^n - 1)}{r - 1} \qquad \Rightarrow \qquad S_n = \frac{1(2^n - 1)}{2 - 1} = \frac{2^n - 1}{1} = 2^n - 1$$

We find that:

$S_5 = 31$
$S_{10} = 1023$
$S_{20} = 1\ 048\ 575$

The sum of the first n terms increases without limit.

Now consider the sum of the first n terms of the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$

$a = 1$ and $r = \frac{1}{2}$

$$S_n = \frac{a(1 - r^n)}{1 - r} \qquad \Rightarrow \qquad S_n = \frac{1\left\{1 - \left(\frac{1}{2}\right)^n\right\}}{1 - \frac{1}{2}} = \frac{1 - \left(\frac{1}{2}\right)^n}{\frac{1}{2}} = 2\left\{1 - \left(\frac{1}{2}\right)^n\right\}$$

We find that:

$S_5 = 1 \cdot 9375$
$S_{10} = 1 \cdot 998...$
$S_{20} = 1 \cdot 999...$

The sum of the first n terms is approaching a limit of 2 as $n \to \infty$.
We say that this geometric series has a **sum to infinity** of 2 and write $S_\infty = 2$.

It is the value of the common ratio r that determines whether or not a geometric series has a sum to infinity. A geometric series has a sum to infinity when $-1 < r < 1$.

In general, the sum of the first n terms of the geometric series with first term a and common ratio r is

$$S_n = \frac{a(1 - r^n)}{1 - r}, \quad r \neq 1.$$

If $-1 < r < 1$, then $r^n \to 0$ as $n \to \infty$ and

$$S_n \to \frac{a(1 - 0)}{1 - r} = \frac{a}{1 - r}.$$

This is the sum to infinity of the geometric series.

$$\boxed{S_\infty = \frac{a}{1 - r}, \quad -1 < r < 1}$$

Worked Example 1

Explain why the geometric series $27 + 18 + 12 + \ldots$ has a sum to infinity and find the value of the sum to infinity.

Solution

$a = 27$ and $r = \frac{18}{27} = \frac{2}{3}$

$-1 < r < 1$, so the geometric series has a sum to infinity.

$$S_\infty = \frac{a}{1-r} = \frac{27}{1-\frac{2}{3}} = \frac{27}{\frac{1}{3}} = 81$$

The sum to infinity is 81.

Worked Example 2

$u_1, u_2, u_3, \ldots$ is a geometric sequence with $u_1 = 48$ and $\sum_{k=1}^{\infty} u_k = 64$.

Find the value of u_4.

Solution

$u_1 = 48 \quad \Rightarrow \quad a = 48$

$$\begin{aligned}
S_\infty = 64 \quad &\Rightarrow \quad \frac{a}{1-r} = 64 \\
&\Rightarrow \quad \frac{48}{1-r} = 64 \\
&\Rightarrow \quad 48 = 64(1-r) \\
&\Rightarrow \quad 48 = 64 - 64r \\
&\Rightarrow \quad 64r = 16 \\
&\Rightarrow \quad r = \frac{16}{64} = \frac{1}{4} \quad \text{[note that } -1 < r < 1 \text{ as expected]}
\end{aligned}$$

$$\begin{aligned}
u_n = ar^{n-1} \quad \Rightarrow \quad u_4 &= ar^3 \\
&= 48 \times \left(\frac{1}{4}\right)^3 \\
&= 48 \times \frac{1}{64} \\
&= \frac{3}{4}
\end{aligned}$$

YOU CAN NOW ATTEMPT THE WORKSHEETS "SEQUENCES AND SERIES 10" AND "SEQUENCES AND SERIES 11".