SYSTEMS OF LINEAR EQUATIONS

The system of linear equations

$$x + 2y = 11$$
$$3x + 4y = 27$$

can be solved using a matrix method as follows.

[A matrix is simply a rectangular array of numbers. You will study matrices in detail in unit 3.]

STEP 1

Write the **coefficients** in a **matrix**.

$$\left(\begin{array}{cc|c} 1 & 2 & 11 \\ 3 & 4 & 27 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \end{array}$$

STEP 2

Make the first entry of row 2 zero by using a suitable combination of row 2 and row 1. Subtracting 3 times row 1 from row 2 will work. This is represented in shorthand as "$R_2 \to R_2 - 3R_1$", meaning "row 2 becomes row 2 minus 3 times row 1".

This gives the new matrix

$$\left(\begin{array}{cc|c} 1 & 2 & 11 \\ 0 & -2 & -6 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \end{array}$$

STEP 3

The solutions can now be found using **back substitution** as follows.

Th equation for row 2 is now $-2y = -6 \quad \Rightarrow \quad y = 3$

The equation for row 1 is $x + 2y = 11 \quad \Rightarrow \quad x + 6 = 11$

$\Rightarrow \quad x = 5$

Hence $x = 5$ and $y = 3$.

Another Example

$$2x + 7y = 6$$
$$-3x + 2y = 16$$

The matrix of coefficients for this system of equations is $\left(\begin{array}{cc|c} 2 & 7 & 6 \\ -3 & 2 & 16 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \end{array}$

The first entry of row 2 will become zero using the **row operation** $R_2 \to 2R_2 + 3R_1$.

This gives the new matrix $\left(\begin{array}{cc|c} 2 & 7 & 6 \\ 0 & 25 & 50 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \end{array}$

$R_2 \quad \Rightarrow \quad 25y = 50$
$\quad \Rightarrow \quad y = 2$

$R_1 \quad \Rightarrow \quad 2x + 7y = 6$
$\quad \Rightarrow \quad 2x + 14 = 6$
$\quad \Rightarrow \quad 2x = -8$
$\quad \Rightarrow \quad x = -4$

Hence $x = -4$ and $y = 2$.

Note

This method of solving a system of linear equations is known as **Gaussian elimination**. The method of Gaussian elimination can also be used to solve a system of three linear equations in three variables.

Example 1

$$x + 2y + 3z = 12$$
$$x + 3y + z = 3$$
$$2x + 5y + 2z = 7$$

STEP 1

Write the coefficients in a matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 12 \\ 1 & 3 & 1 & 3 \\ 2 & 5 & 2 & 7 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

STEP 2

Make the first entry of row 2 zero by using a suitable combination of row 2 and row 1. The row operation $R_2 \to R_2 - R_1$ will work.

Also make the first entry of row 3 zero by using a combination of row 3 and row 1. The row operation $R_3 \to R_3 - 2R_1$ will work.

After performing these row operations, the new matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 12 \\ 0 & 1 & -2 & -9 \\ 0 & 1 & -4 & -17 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

STEP 3

Now make the second entry of row 3 zero by using a combination of row 3 and row 2. The row operation $R_3 \to R_3 - R_2$ will work.

This gives the new matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 12 \\ 0 & 1 & -2 & -9 \\ 0 & 0 & -2 & -8 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

[Note that a combination of row 3 and row 2 must be used at this stage. If a combination of row 3 and row 1 is used to make the second entry of row 3 zero, then the first entry of row 3 will become non-zero.]

STEP 4

The solutions can now be found using back substitution.

$$R_3 \Rightarrow -2z = -8$$
$$\Rightarrow z = 4$$

$$R_2 \Rightarrow y - 2z = -9$$
$$\Rightarrow y - 8 = -9$$
$$\Rightarrow y = -1$$

$$R_1 \Rightarrow x + 2y + 3z = 12$$
$$\Rightarrow x - 2 + 12 = 12$$
$$\Rightarrow x + 10 = 12$$
$$\Rightarrow x = 2$$

Hence $x = 2$, $y = -1$ and $z = 4$.

[These solutions can be checked by substituting them into the original equations:

$$x + 2y + 3z = 2 + 2(-1) + 3(4) = 12 \ \surd$$
$$x + 3y + z = 2 + 3(-1) + 4 = 3 \ \surd$$
$$2x + 5y + 2z = 2(2) + 5(-1) + 2(4) = 7 \ \surd]$$

Example 2

$$x + 2y - z = 3$$
$$2x + 5y + 2z = -3$$
$$4x - 2y + z = 12$$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 2 & -3 \\ 4 & -2 & 1 & 12 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - 4R_1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & -9 \\ 0 & -10 & 5 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$R_3 \to R_3 + 10R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & -9 \\ 0 & 0 & 45 & -90 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \quad \Rightarrow \quad 45z = -90$

$\quad\;\; \Rightarrow \quad z = -2$

$R_2 \quad \Rightarrow \quad y + 4z = -9$

$\quad\;\; \Rightarrow \quad y - 8 = -9$

$\quad\;\; \Rightarrow \quad y = -1$

$R_1 \quad \Rightarrow \quad x + 2y - z = 3$

$\quad\;\; \Rightarrow \quad x - 2 + 2 = 3$

$\quad\;\; \Rightarrow \quad x = 3$

Hence $x = 3$, $y = -1$ and $z = -2$.

YOU CAN NOW ATTEMPT THE WORKSHEET "SYSTEMS OF LINEAR EQUATIONS 1".

Example 3

$$x + y - 2z = 5$$
$$2x + 4y + z = 15$$
$$-3x + 2y + 2z = 14$$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 2 & 4 & 1 & 15 \\ -3 & 2 & 2 & 14 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 + 3R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 0 & 2 & 5 & 5 \\ 0 & 5 & -4 & 29 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$R_3 \to 2R_3 - 5R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 0 & 2 & 5 & 5 \\ 0 & 0 & -33 & 33 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \quad \Rightarrow \quad -33z = 33$
$\quad\;\; \Rightarrow \quad z = -1$

$R_2 \quad \Rightarrow \quad 2y + 5z = 5$
$\quad\;\; \Rightarrow \quad 2y - 5 = 5$
$\quad\;\; \Rightarrow \quad 2y = 10$
$\quad\;\; \Rightarrow \quad y = 5$

$R_1 \quad \Rightarrow \quad x + y - 2z = 5$
$\quad\;\; \Rightarrow \quad x + 5 + 2 = 5$
$\quad\;\; \Rightarrow \quad x + 7 = 5$
$\quad\;\; \Rightarrow \quad x = -2$

Hence $x = -2$, $y = 5$ and $z = -1$.

Example 4

$$2x + y + 4z = -1$$
$$x + 2y - 3z = 8$$
$$3x - 2y + 2z = 10$$

It is convenient to change the order of the equations so that the top-left coefficient is 1.

$$x + 2y - 3z = 8$$
$$2x + y + 4z = -1$$
$$3x - 2y + 2z = 10$$

The matrix of coefficient is $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 2 & 1 & 4 & -1 \\ 3 & -2 & 2 & 10 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - 3R_1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 0 & -3 & 10 & -17 \\ 0 & -8 & 11 & -14 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$R_3 \to 3R_3 - 8R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 0 & -3 & 10 & -17 \\ 0 & 0 & -47 & 94 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \quad \Rightarrow \quad -47z = 94$

$\quad\;\; \Rightarrow \quad z = -2$

$R_2 \quad \Rightarrow \quad -3y + 10z = -17$

$\quad\;\; \Rightarrow \quad -3y - 20 = -17$

$\quad\;\; \Rightarrow \quad -3y = 3$

$\quad\;\; \Rightarrow \quad y = -1$

$R_1 \quad \Rightarrow \quad x + 2y - 3z = 8$

$\quad\;\; \Rightarrow \quad x - 2 + 6 = 8$

$\quad\;\; \Rightarrow \quad x + 4 = 8$

$\quad\;\; \Rightarrow \quad x = 4$

Hence $x = 4$, $y = -1$ and $z = -2$.

Example 5

$$x + 2y = 10$$
$$2x - 3y + z = 3$$
$$x - 4z = 18$$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 10 \\ 2 & -3 & 1 & 3 \\ 1 & 0 & -4 & 18 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - R_1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 10 \\ 0 & -7 & 1 & -17 \\ 0 & -2 & -4 & 8 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$R_3 \to 7R_3 - 2R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 10 \\ 0 & -7 & 1 & -17 \\ 0 & 0 & -30 & 90 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \quad \Rightarrow \quad -30z = 90$
$\quad\quad \Rightarrow \quad z = -3$

$R_2 \quad \Rightarrow \quad -7y + z = -17$
$\quad\quad \Rightarrow \quad -7y - 3 = -17$
$\quad\quad \Rightarrow \quad -7y = -14$
$\quad\quad \Rightarrow \quad y = 2$

$R_1 \quad \Rightarrow \quad x + 2y = 10$
$\quad\quad \Rightarrow \quad x + 4 = 10$
$\quad\quad \Rightarrow \quad x = 6$

Hence $x = 6$, $y = 2$ and $z = -3$.

YOU CAN NOW ATTEMPT THE WORKSHEET "SYSTEMS OF LINEAR EQUATIONS 2".

Worked Example 6

The points (1, 6), (2, 9) and (–3, 34) lie on a parabola with equation $y = ax^2 + bx + c$, where a, b and c are constants.

(a) Write down three equations in a, b and c.
(b) Hence find the equation of the parabola.

Solution

(a) $y = ax^2 + bx + c$

The point (1, 6) lies on the parabola, so substitute $x = 1$ and $y = 6$:

$$6 = a \times 1^2 + b \times 1 + c$$
$$\Rightarrow \quad a + b + c = 6$$

The point (2, 9) lies on the parabola, so substitute $x = 2$ and $y = 9$:

$$9 = a \times 2^2 + b \times 2 + c$$
$$\Rightarrow \quad 4a + 2b + c = 9$$

The point (–3, 34) lies on the parabola, so substitute $x = -3$ and $y = 34$:

$$34 = a \times (-3)^2 + b \times (-3) + c$$
$$\Rightarrow \quad 9a - 3b + c = 34$$

(b)

$$a + b + c = 6$$
$$4a + 2b + c = 9$$
$$9a - 3b + c = 34$$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 4 & 2 & 1 & 9 \\ 9 & -3 & 1 & 34 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$

$$R_2 \to R_2 - 4R_1$$
$$R_3 \to R_3 - 9R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & -3 & -15 \\ 0 & -12 & -8 & -20 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$R_3 \to R_3 - 6R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & -3 & -15 \\ 0 & 0 & 10 & 70 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \quad \Rightarrow \quad 10c = 70$

$\quad\quad \Rightarrow \quad c = 7$

$R_2 \quad \Rightarrow \quad -2b - 3c = -15$

$\quad\quad \Rightarrow \quad -2b - 21 = -15$

$\quad\quad \Rightarrow \quad -2b = 6$

$\quad\quad \Rightarrow \quad b = -3$

$R_1 \quad \Rightarrow \quad a + b + c = 6$

$\quad\quad \Rightarrow \quad a - 3 + 7 = 6$

$\quad\quad \Rightarrow \quad a + 4 = 6$

$\quad\quad \Rightarrow \quad a = 2$

Hence $a = 2$, $b = -3$, $c = 7$ and the equation of the parabola is $y = 2x^2 - 3x + 7$.

YOU CAN NOW ATTEMPT THE WORKSHEET "SYSTEMS OF LINEAR EQUATIONS 2".

REDUNDANT AND INCONSISTENT EQUATIONS

Not all systems of linear equations have a unique solution.

Consider the system of linear equations

$$x + y = 5$$
$$2x + 2y = 10$$

There are infinitely many solutions which satisfy both these equations; for example, $x = 3$, $y = 2$ or $x = 1$, $y = 4$ or ...

The reason that there is not a unique solution in this case is that the second equation gives exactly the same information as the first equation. The second equation is simply double the first equation and is therefore effectively the same equation. So any values of x and y which satisfy $x + y = 5$ will satisfy both equations.

In this context, we say that the second equation is **redundant** as it gives no additional information to the first equation.

Consider now the system of linear equations

$$x + y = 5$$
$$x + y = 10$$

This system of equations has no solution as the second equation contradicts the first equation. It is not possible to find values of x and y such that $x + y = 5$ and $x + y = 10$.

In this context, we say that the system of equations is **inconsistent**.

The following examples illustrate these concepts in the context of a system of three equations in three variables.

Example 1

$$x + 2y + 2z = 11$$
$$x - y + 3z = 8$$
$$4x - y + 11z = 35$$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 1 & -1 & 3 & 8 \\ 4 & -1 & 11 & 35 \end{array}\right) \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$

$$R_2 \rightarrow R_2 - R_1$$
$$R_3 \rightarrow R_3 - 4R_1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & -9 & 3 & -9 \end{array}\right) \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$$R_3 \rightarrow R_3 - 3R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

The row of zeros gives no information and tells us that there is not a unique solution to this system of equations.

[If you look closely at the original equations, you can see that the third equation is actually the first equation plus three times the second equation. Thus the third equation gives no additional information and is redundant.]

Example 2

$$x + 2y + 2z = 11$$
$$2x - y + z = 8$$
$$3x + y + 3z = 18$$

The matrix of coefficients is $\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 2 & -1 & 1 & 8 \\ 3 & 1 & 3 & 18 \end{array}\right) \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - 3R_1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -5 & -3 & -14 \\ 0 & -5 & -3 & -15 \end{array}\right) \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$$R_3 \to R_3 - R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -5 & -3 & -14 \\ 0 & 0 & 0 & -1 \end{array}\right) \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

The third equation gives a contradiction, that $0 = 1$. This means that there is no solution to this system of equations and the system is in fact inconsistent.

[If you look closely at the original equations, you can see that the sum of the first two equations gives the equation $3x + y + 3z = 19$. This is inconsistent with the third equation, which states that $3x + y + 3z = 18$.]