ADVANCED HIGHER MATHEMATICS

INTEGRATION: USE OF STANDARD INTEGRALS

Use the standard integrals

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$$

$$\int (ax+b)^n dx = \frac{1}{a}\cdot\frac{1}{n+1}(ax+b)^{n+1} + C, \quad n \neq -1$$

throughout questions 1 and 2.

1. Find:

(a) $\int 8x^3 dx$ (b) $\int \frac{6}{x^3} dx$ (c) $\int \sqrt{x}dx$

(d) $\int \frac{1}{\sqrt{x}} dx$ (e) $\int (3x^2 - 4x)dx$ (f) $\int \frac{3x^4 + 6}{x^2} dx$

(g) $\int (x^2 + 1)^2 dx$ (h) $\int \frac{x-3}{\sqrt{x}} dx$ (i) $\int \frac{2x^2 - 1}{\sqrt{x}} dx$

(j) $\int (x^2 + 3x)^2 dx$ (k) $\int \frac{3 - x^5}{x^2} dx$ (l) $\int \left(x + \frac{1}{x}\right)^2 dx$

(m) $\int (2x+1)^3 dx$ (n) $\int (3x-1)^4 dx$ (o) $\int (1-2x)^5 dx$

(p) $\int \sqrt{2x+5}dx$ (q) $\int (8x+3)^3 dx$ (r) $\int (5-4x)^2 dx$

(s) $\int \frac{1}{\sqrt{4x+1}} dx$ (t) $\int \sqrt{3x+2}dx$ (u) $\int \frac{1}{(2x-3)^2} dx$

(v) $\int \frac{1}{(3x+7)^3} dx$ (w) $\int \frac{2}{(4x-5)^3} dx$ (x) $\int \frac{1}{\sqrt{5-2x}} dx$

(y) $\int (2x+1)^{\frac{1}{3}} dx$ (z) $\int \frac{1}{(2x-5)^{\frac{3}{2}}} dx$

2. Evaluate:

(a) $\int_0^2 (3+2x)dx$ (b) $\int_1^2 \left(x - 2 + \frac{3}{x^2}\right)dx$ (c) $\int_1^3 \left(x^2 + \frac{1}{x^2}\right)dx$

(d) $\int_0^4 \sqrt{2x+1}dx$ (e) $\int_1^6 \sqrt{x+3}dx$ (f) $\int_0^1 (3x+1)^3 dx$

(g) $\int_1^4 \frac{x^3+1}{\sqrt{x}} dx$ (h) $\int_1^2 \frac{1}{(2x-1)^4} dx$ (i) $\int_1^5 \frac{1}{\sqrt{2x-1}} dx$

(j) $\int_1^3 \left(\frac{2}{x^2} - \frac{5}{x^4}\right)dx$ (k) $\int_1^4 \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)dx$ (l) $\int_1^4 \frac{2x^2+3}{\sqrt{x}} dx$

(m) $\int_0^1 \frac{1}{(4+5x)^2} dx$ (n) $\int_5^{12} (x-4)^{\frac{1}{3}} dx$

Use the standard integral

$$\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + C$$

throughout questions 3, 4 and 5.

3. Find:

(a) $\int e^{2x} dx$ (b) $\int e^{4x+1} dx$ (c) $\int e^{-2x} dx$

(d) $\int e^{\frac{1}{2}x} dx$ (e) $\int 3e^{-x} dx$ (f) $\int 6e^{3x} dx$

(g) $\int 2e^{4x} dx$ (h) $\int \frac{1}{2}e^{6x} dx$ (i) $\int 8e^{-2x} dx$

(j) $\int (2e^{2x} + x)dx$ (k) $\int \left(e^{2x} + \frac{1}{e^{2x}}\right)dx$ (l) $\int \left(3e^{-3x} - \frac{1}{2}e^{2x}\right)dx$

(m) $\int (2e^x - e^{-2x})dx$ (n) $\int e^x(e^x + 1)dx$ (o) $\int e^{2x}(e^x - 1)dx$

(p) $\int (e^x + 1)^2 dx$ (q) $\int (e^x + e^{-x})^2 dx$ (r) $\int \frac{e^x + 1}{e^x} dx$

(s) $\int (8e^{-4x} + 2e^{-6x})dx$ (t) $\int \frac{2}{e^{4x}} dx$ (u) $\int (e^{2x} + 1)^2 dx$

4. Evaluate, expressing each answer in terms of e:

(a) $\int_1^2 (e^x + 1)dx$ (b) $\int_0^1 e^{x+2}dx$ (c) $\int_1^2 e^{2x}dx$

(d) $\int_0^2 e^{-3x}dx$ (e) $\int_0^1 e^{-x}dx$ (f) $\int_0^2 e^{\frac{1}{2}x}dx$

(g) $\int_2^3 \frac{1}{2}e^{-3x}dx$ (h) $\int_0^1 (e^x + e^{-x})dx$

5. Show that $\int_0^1 (e^x - 1)^2 dx = \frac{1}{2}(e^2 - 4e + 5)$

Use the standard integral

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln(ax+b) + C$$

throughout questions 6, 7 and 8.

6. Find:

(a) $\int \frac{1}{x+4}dx$ (b) $\int \frac{1}{2x+1}dx$ (c) $\int \frac{1}{4x+3}dx$

(d) $\int \frac{1}{3x-2}dx$ (e) $\int \frac{3}{x}dx$ (f) $\int \frac{1}{1-x}dx$

(g) $\int \frac{8}{2x+3}dx$ (h) $\int \frac{3}{3x+1}dx$ (i) $\int \frac{2}{1-4x}dx$

(j) $\int \frac{10}{5x+1}dx$ (k) $\int \frac{1}{3-2x}dx$ (l) $\int \frac{1}{2x}dx$

(m) $\int \frac{6}{2x-1}dx$ (n) $\int \frac{6}{1-3x}dx$ (o) $\int \frac{6}{4x+1}dx$

(p) $\int \left(\frac{1}{2x+5} + \frac{1}{x-3}\right)dx$ (q) $\int \left(\frac{2}{3x+1} + \frac{1}{2x-1} - \frac{6}{4x+3}\right)dx$

7. Evaluate, giving each answer correct to 3 decimal places:

(a) $\int_3^7 \frac{1}{x}dx$ (b) $\int_1^3 \frac{1}{2x-1}dx$ (c) $\int_2^5 \frac{1}{x+1}dx$

(d) $\int_0^1 \frac{1}{3x+5}dx$ (e) $\int_1^4 \frac{2}{2x+3}dx$ (f) $\int_4^{10} \frac{5}{x-3}dx$

(g) $\int_2^5 \frac{6}{2x-3}dx$ (h) $\int_4^5 \frac{2}{x-3}dx$ (i) $\int_2^6 \frac{4}{x}dx$

(j) $\int_5^9 \frac{1}{x-3}dx$ (k) $\int_0^1 \frac{1}{3x+2}dx$ (l) $\int_{-4}^0 \frac{1}{1-2x}dx$

(m) $\int_0^1 \frac{8}{4x+3}dx$

8. Show that: (a) $\int_3^5 \frac{1}{x-1}dx = \ln 2$

(b) $\int_1^{13} \frac{1}{2x+1}dx = \ln 3$

Use the standard integrals

$$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + C$$

throughout questions 9, 10 and 11.

9. Find:

(a) $\int \cos 2x\,dx$ (b) $\int \sin 3x\,dx$ (c) $\int \sec^2 2x\,dx$

(d) $\int 6\cos 3x\,dx$ (e) $\int 2\sin 4x\,dx$ (f) $\int \sec^2 3x\,dx$

(g) $\int 4\cos 6x\,dx$ (h) $\int 2\sin 2x\,dx$ (i) $\int \sec^2 4x\,dx$

(j) $\int (\sec^2 x + 1)dx$ (k) $\int (3\cos 3x + 2\sin x)dx$

(l) $\int (8\cos 2x - 4\sin 2x)dx$ (m) $\int (6\sin 2x + 2\cos 4x)dx$

(n) $\int 6\sec^2 2x dx$ (o) $\int \sec^2 \frac{1}{2} x dx$

10. Evaluate:

(a) $\int_0^{\frac{\pi}{2}} \cos x dx$ (b) $\int_0^{\frac{\pi}{3}} \sin x dx$ (c) $\int_0^{\frac{\pi}{4}} \cos 2x dx$

(d) $\int_0^{\frac{\pi}{2}} \sin 2x dx$ (e) $\int_0^{\frac{\pi}{4}} \sec^2 x dx$ (f) $\int_0^{\frac{\pi}{4}} (\sin x + \cos x)dx$

(g) $\int_0^{\frac{\pi}{8}} \cos 4x dx$

11. Show that $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x dx = \frac{2\sqrt{3}}{3}$

ANSWERS

① (a) $2x^4 + C$ (b) $-\frac{3}{x^2} + C$ (c) $\frac{2}{3} x^{\frac{3}{2}} + C$

(d) $2\sqrt{x} + C$ (e) $x^3 - 2x^2 + C$ (f) $x^3 - \frac{6}{x} + C$

(g) $\frac{1}{5} x^5 + \frac{2}{3} x^3 + x + C$ (h) $\frac{2}{3} x^{3/2} - 6\sqrt{x} + C$

(i) $\frac{4}{5} x^{5/2} - 2\sqrt{x} + C$ (j) $\frac{1}{5} x^5 + \frac{3}{2} x^4 + 3x^3 + C$

(k) $-\frac{3}{x} - \frac{1}{4} x^4 + C$ (l) $\frac{1}{3} x^3 + 2x - \frac{1}{x} + C$

(m) $\frac{1}{8}(2x+1)^4 + C$ (n) $\frac{1}{15}(3x-1)^5 + C$ (o) $-\frac{1}{12}(1-2x)^6 + C$

(p) $\frac{1}{3}(2x+5)^{3/2} + C$ (q) $\frac{1}{32}(8x+3)^4 + C$ (r) $-\frac{1}{12}(5-4x)^3 + C$

(s) $\frac{1}{2}\sqrt{4x+1} + C$ (t) $\frac{2}{9}(3x+2)^{3/2} + C$ (u) $-\frac{1}{2(2x-3)} + C$

(v) $-\frac{1}{6(3x+7)^2} + C$ (w) $-\frac{1}{4(4x-5)^2} + C$ (x) $-\sqrt{5-2x} + C$

(y) $\frac{3}{8}(2x+1)^{4/3} + C$ (z) $-\frac{1}{\sqrt{2x-5}} + C$

② (a) 10 (b) 1 (c) $9\frac{1}{3}$ (d) $8\frac{2}{3}$ (e) $12\frac{2}{3}$

(f) $21\frac{1}{4}$ (g) $115\frac{5}{9}$ (h) $\frac{13}{81}$ (i) 2 (j) $-\frac{11}{24}$

(k) $2\frac{2}{3}$ (l) $30\frac{4}{5}$ (m) $\frac{1}{36}$ (n) $11\frac{1}{4}$

3) (a) $\frac{1}{2}e^{2x}+C$ (b) $\frac{1}{4}e^{4x+1}+C$ (c) $-\frac{1}{2}e^{-2x}+C$

(d) $2e^{\frac{1}{2}x}+C$ (e) $-3e^{-x}+C$ (f) $2e^{3x}+C$

(g) $\frac{1}{2}e^{4x}+C$ (h) $\frac{1}{12}e^{6x}+C$ (i) $-4e^{-2x}+C$

(j) $e^{2x}+\frac{1}{2}x^{2}+C$ (k) $\frac{1}{2}e^{2x}-\frac{1}{2}e^{-2x}+C$

(l) $-e^{-3x}-\frac{1}{4}e^{2x}+C$ (m) $2e^{x}+\frac{1}{2}e^{-2x}+C$

(n) $\frac{1}{2}e^{2x}+e^{x}+C$ (o) $\frac{1}{3}e^{3x}-\frac{1}{2}e^{2x}+C$

(p) $\frac{1}{2}e^{2x}+2e^{x}+x+C$ (q) $\frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x}+C$

(r) $x-e^{-x}+C$ (s) $-2e^{-4x}-\frac{1}{3}e^{-6x}+C$

(t) $-\frac{1}{2}e^{-4x}+C$ (u) $\frac{1}{4}e^{4x}+e^{2x}+x+C$

(a) $e^{2}-e+1$ (b) $e^{3}-e^{2}$ (c) $\frac{1}{2}e^{4}-\frac{1}{2}e^{2}$

(d) $\frac{1}{3}-\frac{1}{3}e^{-6}$ (e) $1-e^{-1}$ (f) $2e-2$

g) $\frac{1}{6}e^{-6}-\frac{1}{6}e^{-9}$ (h) $e-e^{-1}$

(a) $\ln(x+4)+C$ (b) $\frac{1}{2}\ln(2x+1)+C$ (c) $\frac{1}{4}\ln(4x+3)+C$

$\frac{1}{3}\ln(3x-2)+C$ (e) $3\ln x+C$ (f) $-\ln(1-x)+C$

$4\ln(2x+3)+C$ (h) $\ln(3x+1)+C$ (i) $-\frac{1}{2}\ln(1-4x)+C$

$2\ln(5x+1)+C$ (k) $-\frac{1}{2}\ln(3-2x)+C$ (l) $\frac{1}{2}\ln x+C$

$3\ln(2x-1)+C$ (n) $-2\ln(1-3x)+C$ (o) $\frac{3}{2}\ln(4x+1)+C$

(p) $\frac{1}{2}\ln(2x+5)+\ln(x-3)+C$

(q) $\frac{2}{3}\ln(3x+1)+\frac{1}{2}\ln(2x-1)-\frac{3}{2}\ln(4x+3)+C$

7 (a) 0.847 (b) 0.805 (c) 0.693 (d) 0.157
(e) 0.788 (f) 9.730 (g) 5.838 (h) 1.386
(i) 4.394 (j) 1.099 (k) 0.305 (l) 1.099
(m) 1.695

9 (a) $\frac{1}{2}\sin 2x+C$ (b) $-\frac{1}{3}\cos 3x+C$
(c) $\frac{1}{2}\tan 2x+C$ (d) $2\sin 3x+C$
(e) $-\frac{1}{2}\cos 4x+C$ (f) $\frac{1}{3}\tan 3x+C$
(g) $\frac{2}{3}\sin 6x+C$ (h) $-\cos 2x+C$
(i) $\frac{1}{4}\tan 4x+C$ (j) $\tan x+x+C$
(k) $\sin 3x-2\cos x+C$ (l) $4\sin 2x+2\cos 2x+C$
(m) $-3\cos 2x+\frac{1}{2}\sin 4x+C$ (n) $3\tan 2x+C$
(o) $2\tan\frac{1}{2}x+C$

10 (a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{2}$ (d) 1 (e) 1
(f) 1 (g) $\frac{1}{4}$

INTEGRATION BY SUBSTITUTION

1. Use the suggested substitution given in brackets to find:

(a) $\int x(x^2+1)^3\,dx \quad (u = x^2+1)$

(b) $\int x^2(x^3+1)^5\,dx \quad (u = x^3+1)$

(c) $\int x(x^2+3)^7\,dx \quad (u = x^2+3)$

(d) $\int x^3(x^4+1)^9\,dx \quad (u = x^4+1)$

(e) $\int x(2x^2+1)^4\,dx \quad (u = 2x^2+1)$

(f) $\int x(1-x^2)^6\,dx \quad (u = 1-x^2)$

(g) $\int x^2(2x^3-1)^5\,dx \quad (u = 2x^3-1)$

(h) $\int xe^{x^2}\,dx \quad (u = x^2)$

(i) $\int \cos x \cdot e^{\sin x}\,dx \quad (u = \sin x)$

(j) $\int \sin^2 x\cos x\,dx \quad (u = \sin x)$

(k) $\int \cos^3 x\sin x\,dx \quad (u = \cos x)$

(l) $\int x\sin(x^2)\,dx \quad (u = x^2)$

(m) $\int x\sqrt{x^2+1}\,dx \quad (u = x^2+1)$

(n) $\int \sec^2 x\tan^4 x\,dx \quad (u = \tan x)$

(o) $\int x(2x^2+1)^3\,dx \quad (u = 2x^2+1)$

(p) $\int x^2e^{x^3}\,dx \quad (u = x^3)$

(q) $\int \sin^4 x\cos x\,dx \quad (u = \sin x)$

(r) $\int x^2(2x^3+1)^4\,dx \quad (u = 2x^3+1)$

(s) $\int x^2\sqrt{x^3+1}\,dx \quad (u = x^3+1)$

(t) $\int \cos x\sqrt{1+\sin x}\,dx \quad (u = 1+\sin x)$

(u) $\int \frac{\ln x}{x}\,dx \quad (u = \ln x)$

(v) $\int x(x^2-3)^5\,dx \quad (u = x^2-3)$

(w) $\int x\sqrt{1-x^2}\,dx \quad (u = 1-x^2)$

(x) $\int x^2(x^3-1)^2\,dx \quad (u = x^3-1)$

(y) $\int xe^{2x^2}\,dx \quad (u = 2x^2)$

(z) $\int 8\cos x\sin^3 x\,dx \quad (u = \sin x)$

2. Use the suggested substitution given in brackets to find:

(a) $\int \frac{x}{x^2+1}\,dx \quad (u = x^2+1)$

(b) $\int \frac{x^2}{x^3+1}\,dx \quad (u = x^3+1)$

(c) $\int \frac{e^x}{e^x+1}\,dx \quad (u = e^x+1)$

(d) $\int \frac{x^3}{1+x^4}\,dx \quad (u = 1+x^4)$

(e) $\int \frac{e^{2x}}{e^{2x}-1}\,dx \quad (u = e^{2x}-1)$

(f) $\int \frac{x}{2x^2+1}\,dx \quad (u = 2x^2+1)$

(g) $\int \frac{\sec^2 x}{\tan x}\,dx \quad (u = \tan x)$

(h) $\int \frac{x}{\sqrt{x^2+4}}\,dx \quad (u = x^2+4)$

(i) $\int \frac{x^2}{\sqrt{x^3+1}}\,dx \quad (u = x^3+1)$

(j) $\int \frac{x}{\sqrt{3x^2-1}}\,dx \quad (u = 3x^2-1)$

(k) $\int \frac{e^{2x}}{\sqrt{e^{2x}+1}}\,dx \quad (u = e^{2x}+1)$

(l) $\int \frac{x}{\sqrt{2x^2+3}}\,dx \quad (u = 2x^2+3)$

(m) $\int \frac{\cos x}{\sqrt{\sin x}}\,dx \quad (u = \sin x)$

(n) $\int \frac{\sin 2x}{\sqrt{\cos 2x}}\,dx \quad (u = \cos 2x)$

(o) $\int \frac{x}{(x^2+1)^4}\,dx \quad (u = x^2+1)$

(p) $\int \frac{x^2}{(x^3+1)^2}\,dx \quad (u = x^3+1)$

(q) $\int \frac{x}{(1-x^2)^3}\,dx \quad (u = 1-x^2)$

(r) $\int \frac{\cos x}{\sin^6 x}\,dx \quad (u = \sin x)$

(s) $\int \frac{\sin x}{\cos^4 x}\,dx \quad (u = \cos x)$

(t) $\int \frac{\cos x}{1+\sin x}\,dx \quad (u = 1+\sin x)$

(u) $\int \frac{\sin x}{1+\cos x}\,dx \quad (u = 1+\cos x)$

(v) $\int \frac{\cos 2x}{1+\sin 2x}\,dx \quad (u = 1+\sin 2x)$

(w) $\int \frac{\sin 4x}{1-\cos 4x}\,dx \quad (u = 1-\cos 4x)$

(x) $\int \frac{4x}{x^2+1}\,dx \quad (u = x^2+1)$

(y) $\int \frac{e^{3x}}{\sqrt{1-e^{3x}}}\,dx \quad (u = 1-e^{3x})$

(z) $\int \frac{\cos 2x}{(1+\sin 2x)^4}\,dx \quad (u = 1+\sin 2x)$

3. (a) By writing $\tan x$ as $\frac{\sin x}{\cos x}$ and using the substitution $u = \cos x$, find $\int \tan x dx$.

(b) Use a similar method to that in part (a) to find $\int \cot x$.

4. Use the suggested substitution given in brackets to find:

(a) $\int x(x+1)^5 dx \quad (u = x+1)$

(b) $\int x(x-1)^3 dx \quad (u = x-1)$

(c) $\int x(x+2)^4 dx \quad (u = x+2)$

(d) $\int x(x-4)^7 dx \quad (u = x-4)$

(e) $\int (x+1)(x+2)^3 dx \quad (u = x+2)$

(f) $\int (x-1)(x+1)^6 dx \quad (u = x+1)$

(g) $\int (x+2)(x-1)^4 dx \quad (u = x-1)$

(h) $\int (x+1)(x+3)^5 dx \quad (u = x+3)$

(i) $\int x\sqrt{x+1}dx \quad (u = x+1)$

(j) $\int x\sqrt{x-2}dx \quad (u = x-2)$

(k) $\int \frac{x}{\sqrt{x+1}}dx \quad (u = x+1)$

(l) $\int \frac{x}{\sqrt{x-4}}dx \quad (u = x-4)$

(m) $\int \frac{x+1}{\sqrt{x+2}}dx \quad (u = x+2)$

(n) $\int \frac{x+1}{\sqrt{x-1}}dx \quad (u = x-1)$

(o) $\int x^3(x^2+1)^4 dx \quad (u = x^2+1)$

(p) $\int x^3(x^2-1)^3 dx \quad (u = x^2-1)$

(q) $\int \sin^4 x \cos^3 x dx \quad (u = \sin x)$

(r) $\int \cos^2 x \sin^3 x dx \quad (u = \cos x)$

(s) $\int \frac{x}{x+1}dx \quad (u = x+1)$

(t) $\int \frac{x}{x-2}dx \quad (u = x-2)$

(u) $\int \frac{x}{(x+1)^2}dx \quad (u = x+1)$

(v) $\int x(2x+1)^3 dx \quad (u = 2x+1)$

(w) $\int x(2x-1)^5 dx \quad (u = 2x-1)$

(x) $\int x(3x+1)^4 dx \quad (u = 3x+1)$

(y) $\int x^2\sqrt{x+1}dx \quad (u = x+1)$

(z) $\int \frac{x^2}{\sqrt{x-1}}dx \quad (u = x-1)$

5. Make use of the substitution $u = \sec x$ to find $\int \sec^3 x \tan x dx$.

6. (a) Make use of the substitution $u = \sin x$ to find $\int \cos^3 x dx$.

(b) Make use of the substitution $u = \cos x$ to find $\int \sin^3 x dx$.

7. Make use of the substitution $u = x^2 + 1$ to find $\int \frac{x^3}{\sqrt{x^2+1}}dx$.

8. Use the suggested substitution given in brackets to evaluate:

(a) $\int_0^1 x(x^2+1)^3 dx \quad (u = x^2+1)$

(b) $\int_0^1 x^2(x^3+1)^4 dx \quad (u = x^3+1)$

(c) $\int_1^2 x(x^2-1)^5 dx \quad (u = x^2-1)$

(d) $\int_0^1 x^3(x^4+1)^2 dx \quad (u = x^4+1)$

(e) $\int_0^{\frac{\pi}{2}} \sin^3 x \cos x dx \quad (u = \sin x)$

(f) $\int_0^{\pi} \sin x \cos^4 x dx \quad (u = \cos x)$

(g) $\int_0^1 \frac{x}{x^2+1}dx \quad (u = x^2+1)$

(h) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x}dx \quad (u = 1+\sin x)$

(i) $\int_0^3 \frac{x}{\sqrt{x^2+16}}dx \quad (u = x^2+16)$

(j) $\int_0^1 x(x+1)^3 dx \quad (u = x+1)$

(k) $\int_0^1 x(x+2)^4 dx \quad (u = x+2)$

(l) $\int_0^8 x\sqrt{x+1}dx \quad (u = x+1)$

(m) $\int_0^{\frac{\pi}{6}} \sin^4 x \cos x dx \quad (u = \sin x)$

(n) $\int_1^6 \frac{x}{\sqrt{x+3}}dx \quad (u = x+3)$

(o) $\int_0^3 \frac{x}{\sqrt{25-x^2}}dx \quad (u = 25-x^2)$

(p) $\int_3^4 x^3(x^2-8)^{\frac{1}{3}} dx \quad (u = x^2-8)$

(q) $\int_0^8 \frac{x+2}{\sqrt{x+1}}dx \quad (u = x+1)$

9. Make use of the substitution $u = e^x - 1$ to show that $\int_1^2 \frac{e^x}{e^x-1}dx = \ln(e+1)$.

10. Use the suggested substitution given in brackets to evaluate:

(a) $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx \quad (x = \sin\theta)$

(b) $\int_0^{1} \frac{1}{\sqrt{4-x^2}}\,dx \quad (x = 2\sin\theta)$

(c) $\int_0^{2} \frac{2}{\sqrt{16-x^2}}\,dx \quad (x = 4\sin\theta)$

(d) $\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx \quad (x = \sin\theta)$

(e) $\int_0^{\frac{3}{2}} \frac{1}{\sqrt{9-x^2}}\,dx \quad (x = 3\sin\theta)$

(f) $\int_0^{\frac{1}{2}} \frac{x+1}{\sqrt{1-x^2}}\,dx \quad (x = \sin\theta)$

(g) $\int_0^{1} \frac{x+2}{\sqrt{4-x^2}}\,dx \quad (x = 2\sin\theta)$

(h) $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx \quad (x = 2\sin\theta)$

(i) $\int_0^{\frac{1}{6}} \frac{1}{\sqrt{1-9x^2}}\,dx \quad \left(x = \frac{1}{3}\sin\theta\right)$

(j) $\int_0^{2} \frac{x+1}{\sqrt{16-x^2}}\,dx \quad (x = 4\sin\theta)$

11. Make use of the substitution $x = 1-\sin\theta$ to evaluate $\int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{2x-x^2}}\,dx$.

12. Make use of a suitable substitution to find:

(a) $\int x(x^2+1)^6\,dx$

(b) $\int x^2(x^3-2)^4\,dx$

(c) $\int x\sqrt{x^2+4}\,dx$

(d) $\int \frac{x}{2x^2+1}\,dx$

(e) $\int \frac{2x^3}{x^4+1}\,dx$

(f) $\int \frac{x^2}{\sqrt{1+x^3}}\,dx$

(g) $\int e^{2x}(e^{2x}+1)^3\,dx$

(h) $\int \frac{\sin x}{2\cos x+3}\,dx$

(i) $\int x^2\sqrt{x^3+1}\,dx$

(j) $\int x(x+1)^4\,dx$

(k) $\int x\sqrt{x-3}\,dx$

(l) $\int \frac{x}{\sqrt{x+2}}\,dx$

(m) $\int x^3(x^2+1)^6\,dx$

(n) $\int x^2\sqrt{x-1}\,dx$

(o) $\int \frac{x^2}{\sqrt{x-2}}\,dx$

13. Make use of a suitable substitution to evaluate:

(a) $\int_0^{1} x(x^2+1)^4\,dx$

(b) $\int_0^{2} x^2\sqrt{x^3+1}\,dx$

(c) $\int_0^{1} \frac{2x}{x^2+1}\,dx$

(d) $\int_0^{4} \frac{x}{\sqrt{x^2+9}}\,dx$

(e) $\int_0^{1} x(x+1)^4\,dx$

(f) $\int_0^{5} x\sqrt{x+4}\,dx$

(g) $\int_5^{10} \frac{x}{\sqrt{x-1}}\,dx$

(h) $\int_0^{\frac{\pi}{4}} \sin 2x\sqrt{1-\cos 2x}\,dx$

(i) $\int_0^{3} \frac{x^2}{\sqrt{x+1}}\,dx$

14. Make use of the substitution $x = 3\sin\theta$ to show that

$$\int_0^{\frac{3}{2}} \frac{x+1}{\sqrt{9-x^2}}\,dx = 3 - \frac{3\sqrt{3}}{2} + \frac{\pi}{6}.$$

ANSWERS

1

(a) $\frac{1}{8}(x^2+1)^4 + C$

(b) $\frac{1}{18}(x^3+1)^6 + C$

(c) $\frac{1}{16}(x^2+3)^8 + C$

(d) $\frac{1}{40}(x^4+1)^{10} + C$

(e) $\frac{1}{20}(2x^2+1)^5 + C$

(f) $-\frac{1}{14}(1-x^2)^7 + C$

(g) $\frac{1}{36}(2x^3-1)^6 + C$

(h) $\frac{1}{2}e^{x^2} + C$

(i) $e^{\sin x} + C$

(j) $\frac{1}{3}\sin^3 x + C$

(k) $-\frac{1}{4}\cos^4 x + C$

(l) $-\frac{1}{2}\cos(x^2) + C$

(m) $\frac{1}{3}(x^2+1)^{3/2} + C$

(n) $\frac{1}{5}\tan^5 x + C$

(o) $\frac{1}{16}(2x^2+1)^4 + C$

(p) $\frac{1}{3}e^{x^3} + C$

(q) $\frac{1}{5}\sin^5 x + C$

(r) $\frac{1}{30}(2x^3+1)^5$

(s) $\frac{2}{9}(x^3+1)^{3/2} + C$

(t) $\frac{2}{3}(1+\sin x)^{3/2} + C$

(u) $\frac{1}{2}(\ln x)^2 + C$

(v) $\frac{1}{12}(x^2-3)^6 + C$

(w) $-\frac{1}{3}(1-x^2)^{3/2} + C$

(x) $\frac{1}{9}(x^3-1)^3 + C$

(y) $\frac{1}{4}e^{2x^2} + C$

(z) $2\sin^4 x + C$

2

(a) $\frac{1}{2}\ln(x^2+1) + C$

(b) $\frac{1}{3}\ln(x^3+1) + C$

(c) $\ln(e^x+1) + C$

(d) $\frac{1}{4}\ln(1+x^4) + C$

(e) $\frac{1}{2}\ln(e^{2x}-1) + C$

(f) $\frac{1}{4}\ln(2x^2+1) + C$

(g) $\ln(\tan x) + C$

(h) $\sqrt{x^2+4} + C$

(i) $\frac{2}{3}\sqrt{x^3+1} + C$

(j) $\frac{1}{3}\sqrt{3x^2-1} + C$

(k) $\sqrt{e^{2x}+1} + C$

(l) $\frac{1}{2}\sqrt{2x^2+3} + C$

(m) $2\sqrt{\sin x} + C$

(n) $-\sqrt{\cos 2x} + C$

(o) $-\frac{1}{6(x^2+1)^3} + C$

(p) $-\frac{1}{3(x^3+1)} + C$

(q) $\frac{1}{4(1-x^2)^2} + C$

(r) $-\frac{1}{5\sin^5 x} + C$

(s) $\frac{1}{3\cos^3 x} + C$

(t) $\ln(1+\sin x) + C$

(u) $-\ln(1+\cos x) + C$

(v) $\frac{1}{2}\ln(1+\sin 2x) + C$

(w) $\frac{1}{4}\ln(1-\cos 4x) + C$

(x) $2\ln(x^2+1) + C$

(y) $-\frac{2}{3}\sqrt{1-e^{3x}} + C$

(z) $-\frac{1}{6(1+\sin 2x)^3} + C$

(3) (a) $-\ln(\cos x) + C$ (b) $\ln(\sin x) + C$

(4) (a) $\frac{1}{7}(x+1)^7 - \frac{1}{6}(x+1)^6 + C$

(b) $\frac{1}{5}(x-1)^5 + \frac{1}{4}(x-1)^4 + C$

(c) $\frac{1}{6}(x+2)^6 - \frac{2}{5}(x+2)^5 + C$

(d) $\frac{1}{9}(x-4)^9 + \frac{1}{2}(x-4)^8 + C$

(e) $\frac{1}{5}(x+2)^5 - \frac{1}{4}(x+2)^4 + C$

(f) $\frac{1}{8}(x+1)^8 - \frac{2}{7}(x+1)^7 + C$

(g) $\frac{1}{6}(x-1)^6 + \frac{3}{5}(x-1)^5 + C$

(h) $\frac{1}{7}(x+3)^7 - \frac{1}{3}(x+3)^6 + C$

(i) $\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$

(j) $\frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2} + C$

(k) $\frac{2}{3}(x+1)^{3/2} - 2\sqrt{x+1} + C$

(l) $\frac{2}{3}(x-4)^{3/2} - 8\sqrt{x-4} + C$

(m) $\frac{2}{3}(x+2)^{3/2} - 2\sqrt{x+2} + C$

(n) $\frac{2}{3}(x-1)^{3/2} + 4\sqrt{x-1} + C$

(o) $\frac{1}{12}(x^2+1)^6 - \frac{1}{10}(x^2+1)^5 + C$

(p) $\frac{1}{10}(x^2-1)^5 + \frac{1}{8}(x^2-1)^4 + C$

(q) $\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$

(r) $-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$

(s) $x + 1 - \ln(x+1) + C$

(t) $x - 2 + 2\ln(x-2) + C$

(u) $\ln(x+1) + \frac{1}{x+1} + C$

(v) $\frac{1}{20}(2x+1)^5 - \frac{1}{16}(2x+1)^4 + C$

(w) $\frac{1}{28}(2x-1)^7 + \frac{1}{24}(2x-1)^6 + C$

(x) $\frac{1}{54}(3x+1)^6 - \frac{1}{45}(3x+1)^5 + C$

(y) $\frac{2}{7}(x+1)^{7/2} - \frac{4}{5}(x+1)^{5/2} + \frac{2}{3}(x+1)^{3/2} + C$

(z) $\frac{2}{5}(x-1)^{5/2} + \frac{4}{3}(x-1)^{3/2} + 2\sqrt{x-1} + C$

(5) $\frac{1}{3}\sec^3 x + C$

(6) (a) $\sin x - \frac{1}{3}\sin^3 x + C$

(b) $-\cos x + \frac{1}{3}\cos^3 x + C$

(7) $\frac{1}{3}(x^2+1)^{3/2} - \sqrt{x^2+1} + C$

(8) (a) $1\frac{7}{8}$ (b) $2\frac{1}{15}$ (c) $60\frac{3}{4}$ (d) $\frac{7}{12}$

(e) $\frac{1}{4}$ (f) $\frac{2}{5}$ (g) $\frac{1}{2}\ln 2 \doteqdot 0.347$

(h) $\ln 2 \doteqdot 0.693$ (i) 1 (j) $2\frac{9}{20}$ (k) $26\frac{13}{30}$

(l) $79\frac{7}{15}$ (m) $\frac{1}{160}$ (n) $6\frac{2}{3}$ (o) 1

(p) $72\frac{3}{14}$ (q) $21\frac{1}{3}$

(10) (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $1 - \frac{\sqrt{3}}{2}$ (e) $\frac{\pi}{6}$

(f) $1 - \frac{\sqrt{3}}{2} + \frac{\pi}{6}$ (g) $2 - \sqrt{3} + \frac{\pi}{3}$ (h) $\frac{\pi}{12}$

(i) $\frac{\pi}{18}$ (j) $4 - 2\sqrt{3} + \frac{\pi}{6}$

(11) $\frac{\pi}{6}$

(12) (a) $\frac{1}{14}(x^2+1)^7 + C$ (b) $\frac{1}{15}(x^3-2)^5 + C$

(c) $\frac{1}{3}(x^2+4)^{3/2} + C$ (d) $\frac{1}{4}\ln(2x^2+1) + C$

(e) $\frac{1}{2}\ln(x^4+1) + C$ (f) $\frac{2}{3}\sqrt{1+x^3} + C$

(g) $\frac{1}{8}(e^{2x}+1)^4 + C$ (h) $-\frac{1}{2}\ln(2\cos x+3) + C$

(i) $\frac{2}{9}(x^3+1)^{3/2} + C$

(j) $\frac{1}{6}(x+1)^6 - \frac{1}{5}(x+1)^5 + C$

(k) $\frac{2}{5}(x-3)^{5/2} + 2(x-3)^{3/2} + C$

(l) $\frac{2}{3}(x+2)^{3/2} - 4\sqrt{x+2} + C$

(m) $\frac{1}{16}(x^2+1)^8 - \frac{1}{14}(x^2+1)^7 + C$

(13) (a) $3\frac{1}{10}$ (b) $5\frac{7}{9}$ (c) $\ln 2 \doteqdot 0.693$ (d) 2

(e) $4\frac{3}{10}$ (f) $33\frac{11}{15}$ (g) $14\frac{2}{3}$ (h) $\frac{1}{3}$ (i) $5\frac{1}{15}$

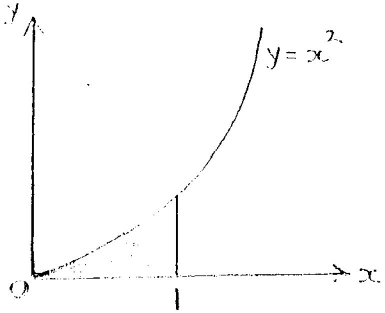
ADVANCED HIGHER MATHEMATICS

VOLUMES OF SOLIDS OF REVOLUTION

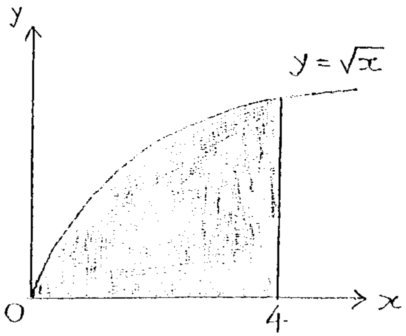
1. Each shaded area below is rotated through one complete revolution about the x-axis.

Find, in terms of π, the volume of the solid of revolution formed in each case.

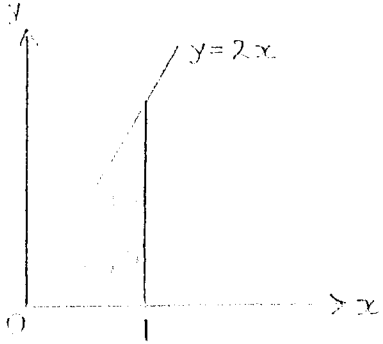
(a)



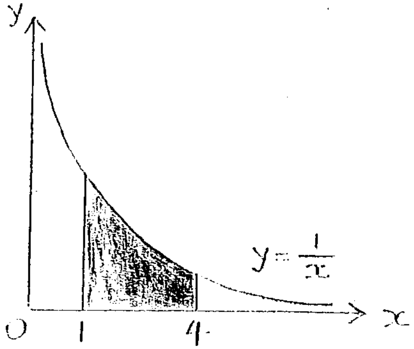
(b)



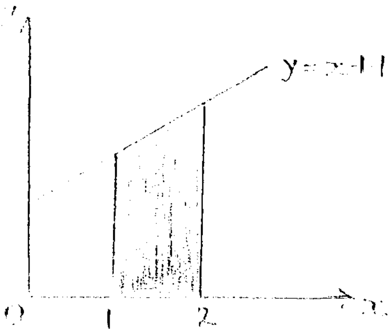
(c)



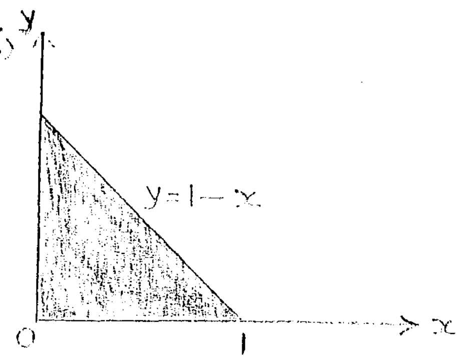
(d)



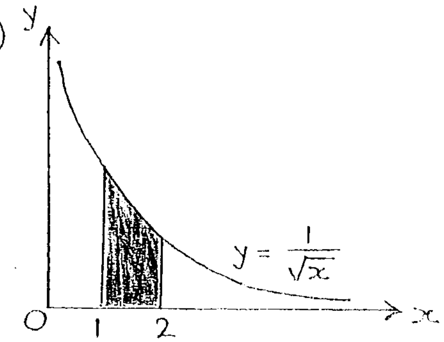
(e)



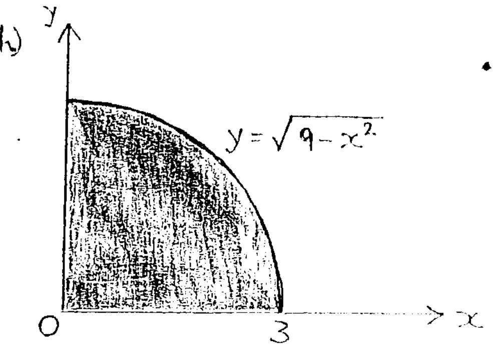
(f)



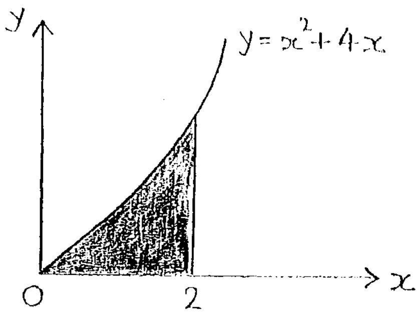
(g)



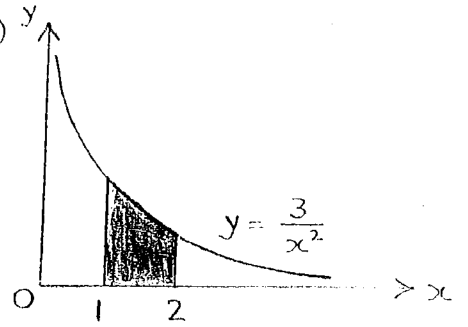
(h)



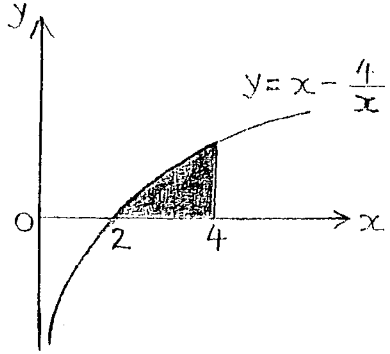
(i)



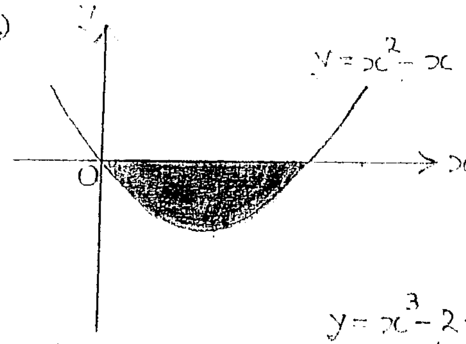
(j)



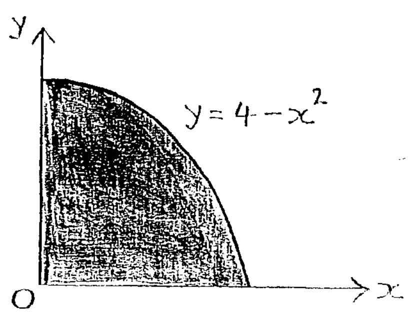
(k)



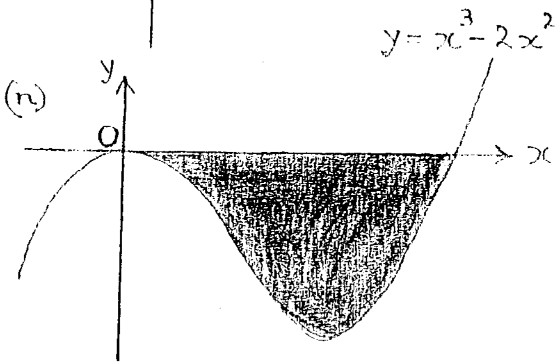
(l)



(m)



(n)



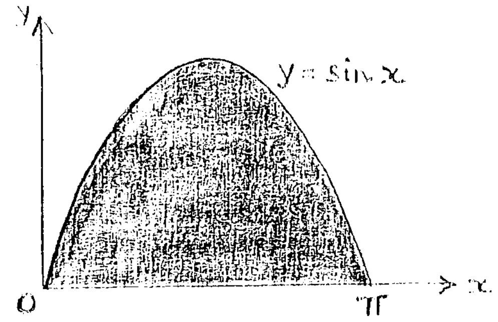
2. (a) Rearrange the identity $\cos 2x = 1 - 2\sin^2 x$ to obtain an identity for $\sin^2 x$ in terms of $\cos 2x$.

Hence show that

$$\int \sin^2 x \, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$$

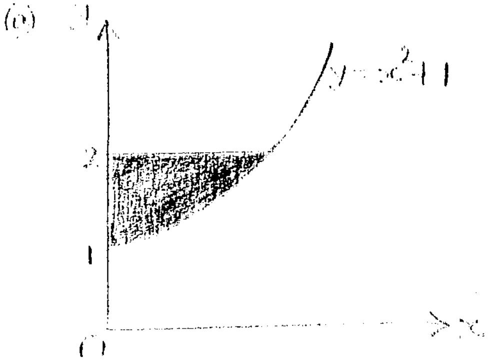
(b) The shaded area in the diagram below is rotated through one complete revolution about the x-axis.

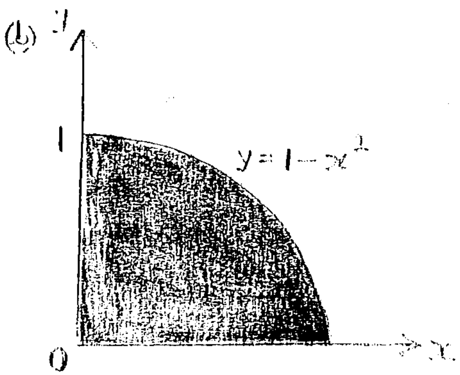
Find, in terms of π, the volume of the solid of revolution formed.

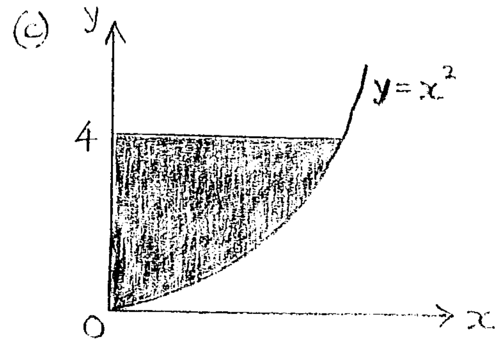


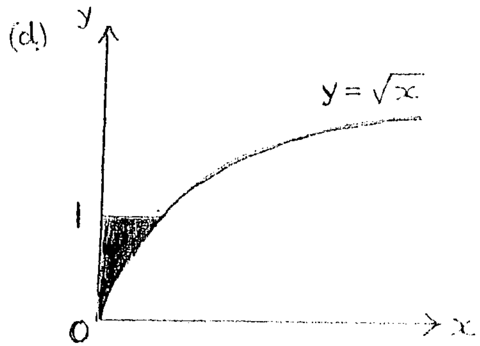
3. Each shaded area below is rotated through one complete revolution about the y-axis.

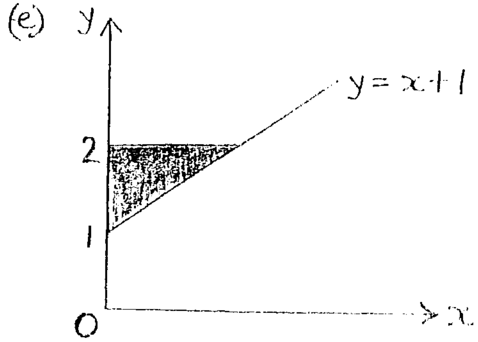
Find, in terms of π, the volume of the solid of revolution formed in each case.

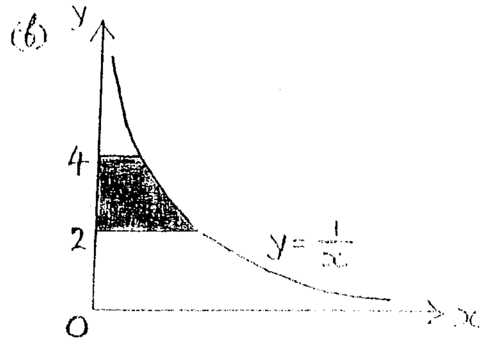


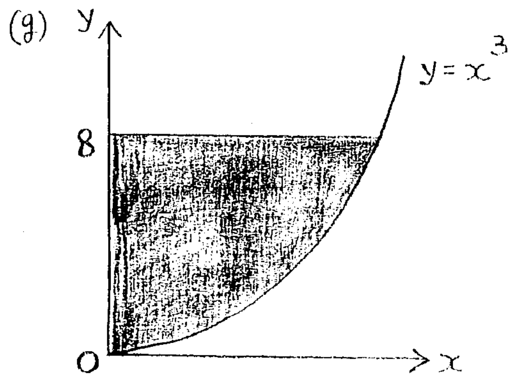


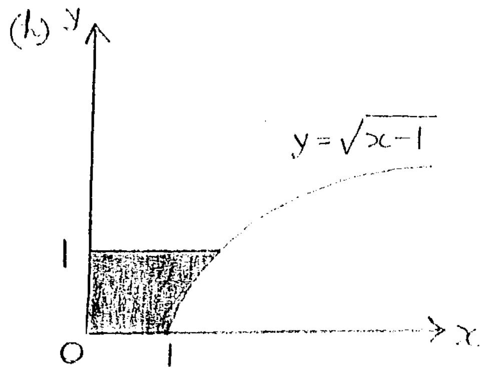












ANSWERS

1. (a) $\frac{\pi}{5}$ cubic units (b) 8π cubic units (c) $\frac{4\pi}{3}$ cubic units

(d) $\frac{3\pi}{4}$ cubic units (e) $\frac{19\pi}{3}$ cubic units (f) $\frac{\pi}{3}$ cubic units

(g) $\pi \ln 2$ cubic units (h) 18π cubic units (i) $\frac{1216\pi}{15}$ cubic units

(j) $\frac{21\pi}{8}$ cubic units (k) $\frac{20\pi}{3}$ cubic units (l) $\frac{\pi}{30}$ cubic units

(m) $\frac{256\pi}{15}$ cubic units (n) $\frac{128\pi}{105}$ cubic units

2. (a) $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ (b) $\frac{\pi^2}{2}$ cubic units

3. (a) $\frac{\pi}{2}$ cubic units (b) $\frac{\pi}{2}$ cubic units (c) 8π cubic units

(d) $\frac{\pi}{5}$ cubic units (e) $\frac{\pi}{3}$ cubic units (f) $\frac{\pi}{4}$ cubic units

(g) $\frac{96\pi}{5}$ cubic units (h) $\frac{28\pi}{15}$ cubic units