# THE BINOMIAL THEOREM

The binomial theorem allows us to expand expressions of the form $(x+y)^n$.

The binomial theorem can be investigated as follows.

(1) Consider $(x+y)^0$.

$$(x+y)^0 = 1$$

We will record this as "1".

(2) Consider $(x+y)^1$.

$$(x+y)^1 = x+y$$

We will record the coefficients of this expansion as "1 1".

(3) Consider $(x+y)^2$.

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y)\\ &= x^2+2xy+y^2\end{aligned}$$

We will record the coefficients of this expansion as "1 2 1".

(4) Consider $(x+y)^3$.

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)^2\\ &= (x+y)(x^2+2xy+y^2)\\ &= x(x^2+2xy+y^2)+y(x^2+2xy+y^2)\\ &= x^3+2x^2y+xy^2+x^2y+2xy^2+y^3\\ &= x^3+3x^2y+3xy^2+y^3\end{aligned}$$

We will record the coefficients of this expansion as "1 3 3 1".

(5) Consider $(x+y)^4$.

$$
\begin{aligned}
(x+y)^4 &= (x+y)(x+y)^3 \\
&= (x+y)(x^3+3x^2y+3xy^2+y^3) \\
&= x(x^3+3x^2y+3xy^2+y^3)+y(x^3+3x^2y+3xy^2+y^3) \\
&= x^4+3x^3y+3x^2y^2+xy^3+x^3y+3x^2y^2+3xy^3+y^4 \\
&= x^4+4x^3y+6x^2y^2+4xy^3+y^4
\end{aligned}
$$

We will record the coefficients in this expansion as "1 4 6 4 1".

## PASCAL'S TRIANGLE

The coefficients in the previous expansions can be recorded in a triangular array as follows:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $(n=0)$ | | | | | 1 | | | | |
| $(n=1)$ | | | | 1 | | 1 | | | |
| $(n=2)$ | | | 1 | | 2 | | 1 | | |
| $(n=3)$ | | 1 | | 3 | | 3 | | 1 | |
| $(n=4)$ | 1 | | 4 | | 6 | | 4 | | 1 |

This triangular array of numbers is known as **Pascal's triangle** and can be extended indefinitely.

Every row starts and ends with 1 and each number inbetween is the sum of the two adjacent numbers in the row above. The coefficients in every row are also symmetrical.

The triangle can be extended to give the coefficients for $n=5$ as follows:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $(n=0)$ | | | | | | 1 | | | | | |
| $(n=1)$ | | | | | 1 | | 1 | | | | |
| $(n=2)$ | | | | 1 | | 2 | | 1 | | | |
| $(n=3)$ | | | 1 | | 3 | | 3 | | 1 | | |
| $(n=4)$ | | 1 | | 4 | | 6 | | 4 | | 1 | |
| $(n=5)$ | 1 | | 5 | | 10 | | 10 | | 5 | | 1 |

Hence

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

**Example 1**

$$\begin{aligned}(x+4)^3 &= x^3 + 3\cdot x^2\cdot 4 + 3\cdot x\cdot 4^2 + 4^3\\ &= x^3 + 12x^2 + 48x + 64\end{aligned}$$

**Example 2**

$$\begin{aligned}(y+3)^5 &= y^5 + 5\cdot y^4\cdot 3 + 10\cdot y^3\cdot 3^2 + 10\cdot y^2\cdot 3^3 + 5\cdot y\cdot 3^4 + 3^5\\ &= y^5 + 15y^4 + 90y^3 + 270y^2 + 405y + 243\end{aligned}$$

**Example 3**

$$\begin{aligned}(2-a)^4 &= 2^4 + 4\cdot 2^3\cdot(-a) + 6\cdot 2^2\cdot(-a)^2 + 4\cdot 2\cdot(-a)^3 + (-a)^4\\ &= 16 - 32a + 24a^2 - 8a^3 + a^4\end{aligned}$$

**Example 4**

$$\begin{aligned}(2x+5)^3 &= (2x)^3 + 3\cdot(2x)^2\cdot 5 + 3\cdot 2x\cdot 5^2 + 5^3\\ &= 8x^3 + 60x^2 + 150x + 125\end{aligned}$$

**Example 5**

$$\begin{aligned}(3c-2d)^4 &= (3c)^4 + 4\cdot(3c)^3\cdot(-2d) + 6\cdot(3c)^2\cdot(-2d)^2 + 4\cdot 3c\cdot(-2d)^3 + (-2d)^4\\ &= 81c^4 - 216c^3d + 216c^2d^2 - 96cd^3 + 16d^4\end{aligned}$$

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**Example 6**

$$\begin{aligned}(x^2+3)^4 &= (x^2)^4 + 4\cdot(x^2)^3\cdot 3 + 6\cdot(x^2)^2\cdot 3^2 + 4\cdot x^2\cdot 3^3 + 3^4\\ &= x^8 + 12x^6 + 54x^4 + 108x^2 + 81\end{aligned}$$

**Example 7**

$$\begin{aligned}\left(x+\frac{1}{x}\right)^5 &= x^5 + 5\cdot x^4\cdot\frac{1}{x} + 10\cdot x^3\cdot\left(\frac{1}{x}\right)^2 + 10\cdot x^2\cdot\left(\frac{1}{x}\right)^3 + 5\cdot x\cdot\left(\frac{1}{x}\right)^4 + \left(\frac{1}{x}\right)^5\\ &= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}\end{aligned}$$

**Example 8**

$$\begin{aligned}\left(x^2-\frac{2}{x}\right)^4 &= (x^2)^4 + 4\cdot(x^2)^3\cdot\left(-\frac{2}{x}\right) + 6\cdot(x^2)^2\cdot\left(-\frac{2}{x}\right)^2 + 6\cdot x^2\cdot\left(-\frac{2}{x}\right)^3 + \left(-\frac{2}{x}\right)^4\\ &= x^8 + 4\cdot x^6\cdot\left(-\frac{2}{x}\right) + 6\cdot x^4\cdot\frac{4}{x^2} + 4\cdot x^2\cdot\left(-\frac{8}{x^3}\right) + \frac{16}{x^4}\\ &= x^8 - 8x^5 + 24x^2 - \frac{32}{x} + \frac{16}{x^4}\end{aligned}$$

**Example 9**

$$\begin{aligned}(2x^2+3)^4 &= (2x^2)^4 + 4\cdot(2x^2)^3\cdot 3 + 6\cdot(2x^2)^2\cdot 3^2 + 4\cdot 2x^2\cdot 3^3 + 3^4\\ &= 16x^8 + 96x^6 + 216x^4 + 216x^2 + 81\end{aligned}$$

**Worked Example 10**

Find the coefficient of $x^3$ in the expansion of $(2x-3)(x+2)^5$.

**Solution**

$$\begin{aligned}(x+2)^5 &= x^5 + 5\cdot x^4\cdot 2 + 10\cdot x^3\cdot 2^2 + 10\cdot x^2\cdot 2^3 + 5\cdot x\cdot 2^4 + 2^5\\ &= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32\end{aligned}$$

$$(2x-3)(x+2)^5 = (2x-3)(x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32)$$

$$\begin{aligned}\text{Term in } x^3 &= 2x\cdot 80x^2 - 3\cdot 40x^3\\ &= 160x^3 - 120x^3\\ &= 40x^3\end{aligned}$$

Hence the coefficient of $x^3$ is 40.

[Note that it is not necessary to expand $(2x-3)(x+2)^5$ fully to obtain the coefficient of one particular term.]

**Worked Example 11**

Expand fully $(x+3)(x-2)^3$.

**Solution**

$$\begin{aligned}(x-2)^3 &= x^3 + 3\cdot x^2\cdot(-2) + 3\cdot x\cdot(-2)^2 + (-2)^3\\ &= x^3 - 6x^2 + 12x - 8\end{aligned}$$

$$\begin{aligned}(x+3)(x-2)^3 &= (x+3)(x^3 - 6x^2 + 12x - 8)\\ &= x(x^3 - 6x^2 + 12x - 8) + 3(x^3 - 6x^2 + 12x - 8)\\ &= x^4 - 6x^3 + 12x^2 - 8x + 3x^3 - 18x^2 + 36x - 24\\ &= x^4 - 3x^3 - 6x^2 + 28x - 24\end{aligned}$$

**YOU CAN NOW ATTEMPT THE WORKSHEET "THE BINOMIAL THEOREM 2".**