

PARTIAL FRACTIONS I

Express in partial fractions:

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|-----|-------------------------------------|-----|------------------------------|-----|-----------------------------|
| 1. | $\frac{2x}{(x-1)(x+1)}$ | 2. | $\frac{10x}{(x-2)(x+3)}$ | 3. | $\frac{4x}{(x+1)(x+5)}$ |
| 4. | $\frac{20}{(x-3)(x+2)}$ | 5. | $\frac{3}{(x-1)(x+2)}$ | 6. | $\frac{5}{(x+2)(x+3)}$ |
| 7. | $\frac{5x-14}{(x-2)(x-3)}$ | 8. | $\frac{3x-2}{(x+2)(x-2)}$ | 9. | $\frac{15-2x}{(2x-1)(x+3)}$ |
| 10. | $\frac{7x+5}{(x-1)(x+3)}$ | 11. | $\frac{-x-7}{(x-3)(x+1)}$ | 12. | $\frac{6x-15}{x(x+3)}$ |
| 13. | $\frac{3x+4}{x(x+2)}$ | 14. | $\frac{-3x-12}{x(x+6)}$ | 15. | $\frac{4-6x}{x(x+4)}$ |
| 16. | $\frac{7x-11}{(5-x)(x+1)}$ | 17. | $\frac{11x+6}{(3x+1)(2x+3)}$ | 18. | $\frac{2x+7}{(2x-3)(4x-1)}$ |
| 19. | $\frac{5x-3}{x^2+x-30}$ | 20. | $\frac{5x-11}{x^2-4x+3}$ | 21. | $\frac{4x+5}{x^2+3x+2}$ |
| 22. | $\frac{4x-10}{x^2-3x}$ | 23. | $\frac{6x-1}{4x^2-1}$ | 24. | $\frac{4x+1}{2x^2+3x+1}$ |
| 25. | $\frac{2x^2-2x-6}{(x+1)(x+2)(x-1)}$ | | | | |

[Hint for question 25:

$$\frac{2x^2-2x-6}{(x+1)(x+2)(x-1)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x-1}$$

$$= \frac{A(x+2)(x-1) + B(x+1)(x-1) + C(x+1)(x+2)}{(x+1)(x+2)(x-1)}$$

$$\Rightarrow 2x^2 - 2x - 6 = A(x+2)(x-1) + B(x+1)(x-1) + C(x+1)(x+2)]$$

ANSWERS

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|-----|---|-----|--|-----|-----------------------------------|
| 1. | $\frac{1}{x-1} + \frac{1}{x+1}$ | 2. | $\frac{4}{x-2} + \frac{6}{x+3}$ | 3. | $\frac{-1}{x+1} + \frac{5}{x+5}$ |
| 4. | $\frac{4}{x-3} + \frac{4}{x+2}$ | 5. | $\frac{1}{x-1} + \frac{1}{x+2}$ | 6. | $\frac{5}{x+2} + \frac{5}{x+3}$ |
| 7. | $\frac{4}{x-2} + \frac{1}{x-3}$ | 8. | $\frac{2}{x+2} + \frac{1}{x-1}$ | 9. | $\frac{4}{2x-1} + \frac{3}{x+3}$ |
| 10. | $\frac{3}{x-1} + \frac{4}{x+3}$ | 11. | $\frac{-5}{2(x-3)} + \frac{3}{2(x+1)}$ | 12. | $\frac{11}{x+3} + \frac{5}{x}$ |
| 13. | $\frac{2}{x} + \frac{1}{x+2}$ | 14. | $\frac{2}{x} + \frac{1}{x+6}$ | 15. | $\frac{1}{x} + \frac{7}{x+4}$ |
| 16. | $\frac{4}{5-x} + \frac{3}{x+1}$ | 17. | $\frac{1}{3x-1} + \frac{3}{2x+3}$ | 18. | $\frac{2}{2x-3} + \frac{3}{4x-1}$ |
| 19. | $\frac{2}{x-5} + \frac{3}{x+6}$ | 20. | $\frac{3}{x-1} + \frac{2}{x-3}$ | 21. | $\frac{1}{x+1} + \frac{3}{x+2}$ |
| 22. | $\frac{10}{3x} + \frac{2}{3(x-3)}$ | 23. | $\frac{1}{2x-1} + \frac{2}{2x+1}$ | 24. | $\frac{3}{x+1} + \frac{2}{2x+1}$ |
| 25. | $\frac{1}{x+1} + \frac{2}{x+2} + \frac{1}{x-1}$ | | | | |

PARTIAL FRACTIONS 2

Express in partial fractions:

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|-----|---------------------------------------|-----|---------------------------------------|
| 1. | $\frac{x^2 - 9x - 2}{(x+1)(x-1)^2}$ | 2. | $\frac{3x^2 - 11x + 5}{(x-2)(x-1)^2}$ |
| 4. | $\frac{2x-7}{(x+1)(x-2)^2}$ | 5. | $\frac{6x^2 + x - 7}{(x-3)(x+2)^2}$ |
| 7. | $\frac{(x+1)(x-3)^2}{16}$ | 8. | $\frac{5x+1}{x(2x+1)^2}$ |
| 10. | $\frac{4x^2 - 8x + 15}{(x+1)(x-2)^2}$ | 11. | $\frac{2x^2 + 7x + 3}{x(x+1)^2}$ |
| 13. | $\frac{3x^2 - x - 1}{(x-2)(x+1)^2}$ | 14. | $\frac{7x^2 + 1}{x^2(x-1)}$ |

ANSWERS

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|-----|--|-----|---|
| 1. | $\frac{x+1}{2} - \frac{x-1}{1} - \frac{(x-1)^2}{5}$ | 2. | $-\frac{x-2}{5} + \frac{x-1}{8} + \frac{(x-1)^2}{3}$ |
| 3. | $\frac{x-2}{1} + \frac{x+3}{3} - \frac{(x+3)^2}{9}$ | 4. | $-\frac{x+1}{1} + \frac{x-2}{1} - \frac{(x-2)^2}{1}$ |
| 5. | $\frac{x-3}{2} + \frac{x+2}{4} - \frac{(x+2)^2}{3}$ | 6. | $-\frac{x}{3} + \frac{x-1}{4} + \frac{(x-1)^2}{4}$ |
| 7. | $\frac{x+1}{1} - \frac{x-3}{1} + \frac{(x-3)^2}{4}$ | 8. | $\frac{x}{1} - \frac{2x+1}{2} + \frac{(2x+1)^2}{3}$ |
| 9. | $\frac{9(1-x)}{7} + \frac{9(x+2)}{7} + \frac{3(x+2)^2}{1}$ | 10. | $\frac{x+1}{3} - \frac{x-2}{1} + \frac{(x-2)^2}{5}$ |
| 11. | $\frac{3}{1} - \frac{x+1}{1} + \frac{(x+1)^2}{2}$ | 12. | $\frac{2(x+1)}{1} - \frac{2(x-1)}{1} + \frac{(x-1)^2}{1}$ |
| 13. | $\frac{x-2}{1} + \frac{x+1}{2} - \frac{(x+1)^2}{1}$ | 14. | $\frac{x-1}{8} - \frac{x}{1} - \frac{x^2}{1}$ |

PARTIAL FRACTIONS 3

1. (a) Show that the quadratic $x^2 + 2x + 2$ is irreducible.
 (b) Hence express $\frac{3x^2 + 2x + 1}{(x + 1)(x^2 + 2x + 2)}$ in partial fractions.
2. (a) Show that the quadratic $x^2 + 2x + 5$ is irreducible.
 (b) Hence express $\frac{8}{(1 - x)(x^2 + 2x + 5)}$ in partial fractions.
3. Express in partial fractions (you must show first that the quadratic factor is irreducible):
- (a) $\frac{4x^2 - 3x + 2}{(x - 1)(x^2 + x + 1)}$ (b) $\frac{x^2 - 10x - 8}{(x - 2)(x^2 + 2x + 4)}$ (c) $\frac{3x - 2}{x(x^2 + 2)}$
 (d) $\frac{4x^2 + 5x + 13}{(x + 1)(x^2 + x + 3)}$ (e) $\frac{2x^2 + 3x + 1}{(x - 1)(x^2 + x + 1)}$ (f) $\frac{x^2 + 2x + 9}{(x - 1)(x^2 + 3)}$

The following question is an actual examination question.

4. (a) Factorise the cubic polynomial $x^3 - x^2 - x - 2$ as the product of a linear factor $l(x)$ and a quadratic factor $q(x)$.
 (b) Show that the quadratic factor $q(x)$ is irreducible.
 (c) Hence express $\frac{5x + 4}{x^3 - x^2 - x - 2}$ in partial fractions.

1. (b) $\frac{2}{x + 1} + \frac{x - 3}{x^2 + 2x + 2}$
2. (b) $\frac{1}{1 - x} + \frac{x + 3}{x^2 + 2x + 5}$
3. (a) $\frac{1}{x - 1} + \frac{3x - 1}{x^2 + x + 1}$ (b) $\frac{2}{x - 2} + \frac{3x}{x^2 + 2x + 4}$
 (c) $-\frac{1}{x} + \frac{x + 3}{x^2 + 2}$ (d) $\frac{4}{x + 1} + \frac{1}{x^2 + x + 3}$
 (e) $\frac{2}{x - 1} + \frac{1}{x^2 + x + 1}$ (f) $\frac{3}{x - 1} + \frac{-2x}{x^2 + 3}$
4. (a) $x^3 - x^2 - x - 2 = (x - 2)(x^2 + x + 1)$
 (b) $\frac{2}{x - 2} + \frac{-2x - 1}{x^2 + x + 1}$

PARTIAL FRACTIONS: MISCELLANEOUS QUESTIONS

ANSWERS

1. Express in partial fractions (remember to ensure that the denominator of each fraction is factorised fully):

(a) $\frac{8x-3}{x(x-1)}$ (b) $\frac{3x}{(x-2)(x+1)}$ (c) $\frac{4}{(x+1)(x-1)^2}$

(d) $\frac{4x-9}{(x-2)(x-3)}$ (e) $\frac{5x+1}{(x-1)(x^2+x+1)}$ (f) $\frac{3x^2+1}{x(x+1)^2}$

(g) $\frac{3x+4}{x^2+4x}$ (h) $\frac{x+10}{x^2-x-12}$ (i) $\frac{x^2-2x+10}{(x+2)(x-1)^2}$

(j) $\frac{8}{(1-x)(x^2+2x+5)}$ (k) $\frac{x+16}{2x^2+x-6}$ (l) $\frac{x+1}{x^2(x-1)}$

(m) $\frac{6x-1}{4x^2-1}$ (n) $\frac{2x^2+7x+3}{x^3+2x^2+x}$ (o) $\frac{x^2-9x-2}{(x-1)(x^2-1)}$

*2. Express $\frac{9-5x}{(x-1)(x-2)(x-3)}$ in partial fractions.

Hint: Note that the denominator contains distinct linear factors.

$$\frac{9-5x}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$= \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

*3. (a) Prove that x^2+2 does not factorise into the product of two linear factors with real coefficients.
 (b) Express $\frac{3x-2}{x(x^2+2)}$ in partial fractions.

*4. (a) Factorise x^3+2x^2+4x+3 into the product of a linear factor and a quadratic factor, and prove that the quadratic factor is *irreducible*.

(b) Express $\frac{4x^2+2x+7}{x^3+2x^2+4x+3}$ in partial fractions

*5. (a) Factorise x^3-3x^2+3x-2 into the product of a linear factor and a quadratic factor, and prove that the quadratic factor is *irreducible*.

(b) Express $\frac{4x^2-4x+1}{x^3-3x^2+3x-2}$ in partial fractions.

1. (a) $\frac{3}{x} + \frac{5}{x-1}$ (b) $\frac{2}{x-2} + \frac{1}{x+1}$ (c) $\frac{1}{x+1} - \frac{1}{x-1} + \frac{2}{(x-1)^2}$

(d) $\frac{1}{x-2} + \frac{3}{x-3}$ (e) $\frac{2}{x-1} + \frac{-2x+1}{x^2+x+1}$ (f) $\frac{1}{x} + \frac{2}{x+1} + \frac{4}{(x+1)^2}$

(g) $\frac{1}{x} + \frac{2}{x+4}$ (h) $\frac{2}{x-4} - \frac{1}{x+3}$ (i) $\frac{2}{x+2} - \frac{1}{x-1} + \frac{3}{(x-1)^2}$

(j) $\frac{1}{1-x} + \frac{x+3}{x^2+2x+5}$ (k) $\frac{5}{2x-3} - \frac{2}{x+2}$ (l) $\frac{2}{x-1} - \frac{2}{x} - \frac{1}{x^2}$

(m) $\frac{1}{2x-1} + \frac{2}{2x+1}$ (n) $\frac{3}{x} - \frac{1}{x+1} + \frac{2}{(x+1)^2}$ (o) $\frac{2}{x+1} - \frac{1}{x-1} - \frac{5}{(x-1)^2}$

2. $\frac{2}{x-1} + \frac{1}{x-2} - \frac{3}{x-3}$

3. (a) The discriminant of x^2+2 is -8 , which is less than zero, hence x^2+2 does not factorise.

(b) $\frac{1}{x} + \frac{x+3}{x^2+2}$

4. (a) $x^3+2x^2+4x+3 = (x+1)(x^2+x+3)$
 The discriminant of x^2+x+3 is -11 , which is less than zero, hence x^2+x+3 does not factorise.

(b) $\frac{3}{x+1} + \frac{x-2}{x^2+x+3}$

5. (a) $x^3-3x^2+3x-2 = (x-2)(x^2-x+1)$
 The discriminant of x^2-x+1 is -3 , which is less than zero, hence x^2-x+1 does not factorise.

(b) $\frac{3}{x-2} + \frac{x+1}{x^2-x+1}$

ALGEBRAIC LONG DIVISION

Use algebraic long division to express each of the improper rational functions below as the sum of a polynomial function and a proper rational function.

1. $\frac{x^2 + 3x + 5}{x + 2}$
2. $\frac{x^2 - 2x + 4}{x + 3}$
3. $\frac{x^2 + 3x - 5}{x - 2}$
4. $\frac{x^2 - 2x - 1}{x + 1}$
5. $\frac{x^2 + x - 3}{x - 1}$
6. $\frac{x^2 + 2x - 7}{x + 4}$
7. $\frac{2x^2 + 7x + 2}{x + 3}$
8. $\frac{2x^2 - x + 3}{x + 1}$
9. $\frac{2x^2 - 5x + 7}{x - 1}$
10. $\frac{3x^2 - 5x + 1}{x - 4}$
11. $\frac{2x^2 + 5x + 1}{x + 2}$
12. $\frac{x^2 + x + 3}{x^2 + x - 1}$
13. $\frac{3x^2 + 4x + 5}{x^2 + x + 1}$
14. $\frac{x^2 - x + 1}{x^2 + x - 2}$
15. $\frac{2x^2 + 3x + 1}{x^2 + x + 2}$
16. $\frac{x^3 + 3x^2 + 4x - 5}{x^2 + 1}$
17. $\frac{x^2}{x^2 - x + 2}$
18. $\frac{x^3 + 1}{x^2 + 2}$
19. $\frac{x^2 + 3}{x^2 - 4}$
20. $\frac{x^3 - 2x^2 - 6x - 11}{x - 4}$
21. $\frac{x^4 + 3x^3 + 2x^2 - 3}{x^2 + 2x}$
22. $\frac{x^4 - 2x + 5}{x^2 + 4}$
23. $\frac{x^3 + 3x^2 + 7}{x^2 - 2x}$
24. $\frac{x^3 + 4x^2 - x + 2}{x^2 + x}$
25. $\frac{3x^3 + 7x - 1}{x^2 + 3}$

ANSWERS

1. $x + 1 + \frac{3}{x + 2}$
2. $x - 5 + \frac{19}{x + 3}$
3. $x + 5 + \frac{5}{x - 2}$
4. $x - 3 + \frac{2}{x + 1}$
5. $x + 2 - \frac{1}{x - 1}$
6. $x - 2 + \frac{1}{x + 4}$
7. $2x + 1 - \frac{1}{x + 3}$
8. $2x - 3 + \frac{6}{x + 1}$
9. $2x - 3 + \frac{4}{x - 1}$
10. $3x + 7 + \frac{29}{x - 4}$
11. $2x + 1 - \frac{1}{x + 2}$
12. $1 + \frac{4}{x^2 + x - 1}$
13. $3 + \frac{x + 2}{x^2 + x + 1}$
14. $1 + \frac{-2x + 3}{x^2 + x - 2}$
15. $2 + \frac{x - 3}{x^2 + x + 2}$
16. $x + 3 + \frac{3x - 8}{x^2 + 1}$
17. $1 + \frac{x - 2}{x^2 - x + 2}$
18. $x^2 - 2x + 4 - \frac{7}{x + 2}$
19. $1 + \frac{7}{x^2 - 4}$
20. $x^2 + 2x + 2 - \frac{3}{x - 4}$
21. $x^2 + x - \frac{3}{x^2 + 2x}$
22. $x^2 - 4 + \frac{-2x + 21}{x^2 + 4}$
23. $x + 5 + \frac{10x + 7}{x^2 - 2x}$
24. $x + 3 + \frac{-4x + 2}{x^2 + x}$
25. $3x + \frac{-2x - 1}{x^2 + 3}$

PARTIAL FRACTIONS FOR IMPROPER RATIONAL FUNCTIONS

Using algebraic long division first, express each of the improper rational functions below as the sum of a polynomial function and partial fractions.

1. $\frac{x^2 - x + 6}{x^2 + x - 2}$
2. $\frac{x^3 - x^2 - 5x + 1}{x^2 - 2x - 3}$
3. $\frac{x^3 - 5x^2 + 11x - 12}{x^2 - 5x + 6}$
4. $\frac{x^2}{x^2 - 4}$
5. $\frac{x^3 + 4x^2 - x + 2}{x^2 + x}$
6. $\frac{x^2 + 3}{x^2 - 1}$
7. $\frac{(x+3)(x-1)}{(x+2)(x+1)}$
8. $\frac{x^3 + 2x^2 - 5x - 6}{x^2 - 4x + 3}$

1. $1 - \frac{x+2}{4} + \frac{x-1}{2}$
2. $x+1 + \frac{x-3}{1} - \frac{x+1}{1}$
3. $x + \frac{x-3}{3} + \frac{x-2}{2}$
4. $1 + \frac{x-2}{1} - \frac{x+2}{1}$
5. $x+3 + \frac{x}{2} - \frac{x+1}{6}$
6. $1 + \frac{x-1}{2} - \frac{x+1}{2}$
7. $1 + \frac{x}{3} - \frac{x+2}{4}$
8. $x+6 + \frac{x-3}{12} + \frac{x-1}{4}$

ANSWERS

HOMEWORK ON PARTIAL FRACTIONS

1. Express in partial fractions:

- (a) $\frac{4x-1}{(x-1)(x+2)}$ (b) $\frac{x+14}{(x-4)(x+2)}$ (c) $\frac{x^2+6x-3}{x(x-1)^2}$
- (d) $\frac{5-2x}{x(1-x)}$ (e) $\frac{3x^2+4x+9}{(x-1)(x^2+2x+5)}$ (f) $\frac{3x^2-x-1}{(x-2)(x+1)^2}$
- (g) $\frac{2x+5}{(2x-1)(x+1)}$ (h) $\frac{3x^2-5x+3}{x^2(1-x)}$ (i) $\frac{5x-3}{x^2+x-30}$
- (j) $\frac{x^2+2x+9}{(x-1)(x^2+3)}$

2. Express each of the improper rational functions below as the sum of a polynomial function and partial fractions.

- (a) $\frac{x^2+2x-6}{x^2+x-2}$ (b) $\frac{x^3+x}{x^4+2x^2-2x+1}$

3. Express $\frac{x+3}{x(x-1)(x+1)}$ in partial fractions.

4. (a) Express the cubic polynomial x^3-2x^2-x-6 as the product of a linear factor and a quadratic factor, and prove that the quadratic factor is irreducible.

(b) Express $\frac{6x-4}{x^3-2x^2-x-6}$ in partial fractions.