ALGEBRAIC OPERATIONS

PARTIAL FRACTIONS

Recall that fractions can be added by using a **common denominator**.

Example

$$\begin{aligned}\frac{2}{x+1}+\frac{3}{x-2} &= \frac{2(x-2)}{(x+1)(x-2)}+\frac{3(x+1)}{(x+1)(x-2)}\\ &= \frac{2(x-2)+3(x+1)}{(x+1)(x-2)}\\ &= \frac{2x-4+3x+3}{(x+1)(x-2)}\\ &= \frac{5x-1}{(x+1)(x-2)}\end{aligned}$$

[This working can be shortened with practice.]

Hence $\dfrac{5x-1}{(x+1)(x-2)}=\dfrac{2}{x+1}+\dfrac{3}{x-2}$.

Writing $\dfrac{5x-1}{(x+1)(x-2)}$ as $\dfrac{2}{x+1}+\dfrac{3}{x-2}$ is known as writing $\dfrac{5x-1}{(x+1)(x-2)}$ in **partial fractions**.

Partial fractions can be used to aid differentiation and integration and can also be used to find the sum of certain series.

Note that the denominator of $\dfrac{5x-1}{(x+1)(x-2)}$ contains **distinct linear factors** and that the partial fractions are of the form $\dfrac{A}{x+1}+\dfrac{B}{x-2}$ for some constants A and B. This is true in general.

Worked Example 1

Express $\frac{x+7}{(x-2)(x+1)}$ in partial fractions.

Solution

Note that the denominator contains **distinct linear factors.**

$$\frac{x+7}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$
$$= \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

Hence $x+7 = A(x+1)+B(x-2)$.

To eliminate the first bracket and find the value of *B*, put $x = -1$:

$$6 = A(0)+B(-3)$$
$$\Rightarrow \quad -3B = 6$$
$$\Rightarrow \quad B = -2$$

To eliminate the second bracket and find the value of *A*, put $x = 2$:

$$9 = A(3)+B(0)$$
$$\Rightarrow \quad 3A = 9$$
$$\Rightarrow \quad A = 3$$

Hence $\frac{x+7}{(x-2)(x+1)} = \frac{3}{x-2} - \frac{2}{x+1}$.

[*Check*: $\frac{3}{x-2} - \frac{2}{x+1} = \frac{3(x+1)-2(x-2)}{(x-2)(x+1)}$
$= \frac{3x+3-2x+4}{(x-2)(x+1)}$
$= \frac{x+7}{(x-2)(x+1)}$]

Worked Example 2

Express $\dfrac{4-3x}{2x^2+3x-2}$ in partial fractions.

Solution

We must first factorise the denominator:

$$\frac{4-3x}{2x^2+3x-2} = \frac{4-3x}{(2x-1)(x+2)}$$

Note that the denominator contains **distinct linear factors.**

$$\frac{4-3x}{(2x-1)(x+2)} = \frac{A}{2x-1} + \frac{B}{x+2}$$
$$= \frac{A(x+2)+B(2x-1)}{(2x-1)(x+2)}$$

Hence $4-3x = A(x+2)+B(2x-1)$.

To eliminate the first bracket and find the value of B, put $x=-2$:

$$10 = A(0)+B(-5)$$
$$\Rightarrow \quad -5B = 10$$
$$\Rightarrow \quad B = -2$$

To eliminate the second bracket and find the value of A, put $x = \dfrac{1}{2}$:

$$2\frac{1}{2} = A\left(2\frac{1}{2}\right) + B(0)$$
$$\Rightarrow \quad 2\frac{1}{2}A = 2\frac{1}{2}$$
$$\Rightarrow \quad A = 1$$

Hence $\dfrac{4-3x}{2x^2+3x-2} = \dfrac{1}{2x-1} - \dfrac{2}{x+2}$.

YOU CAN NOW ATTEMPT THE WORKSHEET "PARTIAL FRACTIONS 1".

If the denominator contains a **repeated linear factor**, more than one partial fraction must be included for this factor, as illustrated in the example below.

Worked Example 3

Express $\dfrac{x^2 - 7x + 9}{(x+2)(x-1)^2}$ in partial fractions.

Solution

Note that the denominator contains a **repeated linear factor**.

$$\frac{x^2 - 7x + 9}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$
$$= \frac{A(x-1)^2 + B(x+2)(x-1) + C(x+2)}{(x+2)(x-1)^2}$$

Hence $x^2 - 7x + 9 = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$.

Put $x = 1$ $\Rightarrow$ $3 = A(0)^2 + B(3)(0) + C(3)$
$\Rightarrow$ $3C = 3$
$\Rightarrow$ $C = 1$

Put $x = -2$ $\Rightarrow$ $27 = A(-3)^2 + B(0)(-3) + C(0)$
$\Rightarrow$ $9A = 27$
$\Rightarrow$ $A = 3$

To find the value of B, we must substitute a third number for x. It is convenient to use a simple number, such as $x = 0$.

Put $x = 0$ $\Rightarrow$ $9 = A(-1)^2 + B(2)(-1) + C(2)$
$\Rightarrow$ $A - 2B + 2C = 9$
$\Rightarrow$ $3 - 2B + 2 = 9$
$\Rightarrow$ $5 - 2B = 9$
$\Rightarrow$ $-2B = 4$
$\Rightarrow$ $B = -2$

Hence $\dfrac{x^2 - 7x + 9}{(x+2)(x-1)^2} = \dfrac{3}{x+2} - \dfrac{2}{x-1} + \dfrac{1}{(x-1)^2}$.

YOU CAN NOW ATTEMPT THE WORKSHEET "PARTIAL FRACTIONS 2".

If the denominator contains an **irreducible quadratic factor** $q(x)$, the partial fraction corresponding to this factor is of the form $\frac{Bx+C}{q(x)}$, as illustrated in the example below.

[Note that an irreducible quadratic factor cannot be expressed as the product of two linear factors with real coefficients; the discriminant can be used to verify that a quadric factor is indeed irreducible.]

Worked Example 4

Express $\frac{3x^2+2x+1}{(x+1)(x^2+2x+2)}$ in partial fractions.

Solution

We must first use the discriminant to verify that the quadratic factor x^2+2x+2 is **irreducible**.

For x^2+2x+2: $\quad a=1,\ b=2,\ c=2$

$$b^2-4ac = 2^2-4\times1\times2 = -4$$

$b^2-4ac<0$, so x^2+2x+2 is irreducible.

$$\frac{3x^2+2x+1}{(x+1)(x^2+2x+2)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+2x+2} = \frac{A(x^2+2x+2)+(Bx+C)(x+1)}{(x+1)(x^2+2x+2)}$$

Hence $3x^2+2x+1 = A(x^2+2x+2)+(Bx+C)(x+1)$.

Put $x=-1$ $\quad\Rightarrow\quad 2=A(1)+0$

$\Rightarrow\quad A=2$

To find the values of B and C, we must substitute two other numbers for x. It is convenient to use simple numbers, such as $x=0$ and $x=1$.

Put $x=0$ $\quad\Rightarrow\quad 1=A(2)+(C)(1)$

$\Rightarrow\quad 2A+C=1$

$\Rightarrow\quad 4+C=1$

$\Rightarrow\quad C=-3$

Put $x = 1$ $\Rightarrow$ $6 = A(5) + (B + C)(2)$

$\Rightarrow$ $5A + 2B + 2C = 6$

$\Rightarrow$ $10 + 2B - 6 = 6$

$\Rightarrow$ $4 + 2B = 6$

$\Rightarrow$ $2B = 2$

$\Rightarrow$ $B = 1$

Hence $\dfrac{3x^2 + 2x + 1}{(x+1)(x^2 + 2x + 2)} = \dfrac{2}{x+1} + \dfrac{x-3}{x^2 + 2x + 2}$.

YOU CAN NOW ATTEMPT THE WORKSHEET "PARTIAL FRACTIONS 3".

SUMMARY OF PARTIAL FRACTIONS

The type of partial fractions depends on the nature of the factors in the denominator. Always make sure that the denominator is fully factorised before determining the type of partial fractions.

TYPE 1

Denominator contains **distinct linear factors**:

Example: $$\frac{3x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

TYPE 2

Denominator contains a **repeated linear factor**:

Example: $$\frac{x^2+2x+10}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

TYPE 3

Denominator contains an **irreducible quadratic factor**:

Example: $$\frac{5x+1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

[You must remember to use the discriminant to verify that x^2+x+1 is irreducible to justify using these partial fractions.]

YOU CAN NOW ATTEMPT THE WORKSHEET "PARTIAL FRACTIONS; MISCELLANEOUS QUESTIONS".

ALGEBRAIC LONG DIVISION

An improper numerical fraction can be expressed as the sum of a whole number and a proper fraction.

Example: $\frac{14}{5} = 2 + \frac{4}{5}$

The algebraic fraction $\frac{x+1}{x^2+2}$ is an example of a proper rational function, since the degree of the numerator is less than the degree of the denominator.

The algebraic fraction $\frac{x^2+2}{x+1}$ is an example of an **improper rational function**, since the degree of the numerator is greater than or equal to the degree of the denominator.

An improper rational function can be expressed as the sum of a polynomial and a proper rational function using **algebraic long division**.

Worked Example 1

Express $\frac{x^2+2x+4}{x+1}$ as the sum of a polynomial function and a proper rational function.

Solution

Set up the division as below.

$$x+1 \overline{\big)\, x^2 + 2x + 4}$$

Consider the first term of the divisor $x+1$ (x) and the first term of x^2+2x+4 (x^2).
To change x into x^2 you multiply by x.
You can now write x in the appropriate position at the top of the division as below.

$$\begin{array}{r|l} & x \\ \hline x+1 & x^2 + 2x + 4 \end{array}$$

Now multiply the divisor $(x+1)$ by the factor x and write the answer as below.

$$\begin{array}{r|l} & x \\ \hline x+1 & x^2 + 2x + 4 \\ & x^2 + \ \ x \end{array}$$

Now subtract $(x^2 + x)$ from $(x^2 + 2x + 4)$ as below.

$$\begin{array}{r|l} & \quad x \\ \hline x+1 & x^2 + 2x + 4 \\ & x^2 + x \\ \cline{2-2} & \quad x + 4 \end{array}$$

The steps carried out so far are then repeated.

Consider the first term of the divisor $x + 1$ (x) and the first term of $x + 4$ (x).
To change x into x you multiply by 1.
You can now write 1 in the appropriate position at the top of the division as below.

$$\begin{array}{r|l} & \quad x + 1 \\ \hline x+1 & x^2 + 2x + 4 \\ & x^2 + x \\ \cline{2-2} & \quad x + 4 \end{array}$$

Now multiply the divisor $(x + 1)$ by the factor 1 and write the answer as below.

$$\begin{array}{r|l} & \quad x + 1 \\ \hline x+1 & x^2 + 2x + 4 \\ & x^2 + x \\ \cline{2-2} & \quad x + 4 \\ & \quad x + 1 \end{array}$$

Now subtract $(x + 1)$ from $(x + 4)$ as below.

$$\begin{array}{r|l} & \quad x + 1 \\ \hline x+1 & x^2 + 2x + 4 \\ & x^2 + x \\ \cline{2-2} & \quad x + 4 \\ & \quad x + 1 \\ \cline{2-2} & \qquad\; 3 \end{array}$$

The process cannot now be continued and the answer can be read from the division as below.

$$\frac{x^2 + 2x + 4}{x + 1} = x + 1 + \frac{3}{x + 1}$$

Worked Example 2

Express $\dfrac{x^3 + 4x^2 - x + 2}{x^2 + x}$ as the sum of a polynomial and a proper rational function.

Solution

$$
\begin{array}{r|l}
 & \quad\quad\quad\quad x + 3 \\
\hline
x^2 + x & x^3 + 4x^2 - x + 2 \\
 & x^3 + x^2 \\
\hline
 & \quad 3x^2 - x + 2 \\
 & \quad 3x^2 + 3x \\
\hline
 & \quad\quad -4x + 2
\end{array}
$$

Hence $\dfrac{x^3 + 4x^2 - x + 2}{x^2 + x} = x + 3 + \dfrac{-4x + 2}{x^2 + x}$.

Worked Example 3

Express $\dfrac{x^3 - 2x + 5}{x^2 + 2x - 3}$ as the sum of a polynomial and a proper rational function.

Solution

$$
\begin{array}{r|l}
 & \quad\quad\quad\quad x - 2 \\
\hline
x^2 + 2x - 3 & x^3 + 0x^2 - 2x + 5 \\
 & x^3 + 2x^2 - 3x \\
\hline
 & \quad -2x^2 + x + 5 \\
 & \quad -2x^2 - 4x + 6 \\
\hline
 & \quad\quad 5x - 1
\end{array}
$$

Hence $\dfrac{x^3 - 2x + 5}{x^2 + 2x - 3} = x - 2 + \dfrac{5x - 1}{x^2 + 2x - 3}$.

Worked Example 4

Express $\dfrac{3x^3 - 2x^2 + 6}{x^2 + 4}$ as the sum of a polynomial and a proper rational function.

Solution

$$
\begin{array}{r|l}
 & \quad\quad\quad\quad\; 3x - 2 \\
\hline
x^2 + 4 & 3x^3 - 2x^2 + 0x + 6 \\
 & 3x^3 \quad\quad\; + 12x \\
\hline
 & \quad\; -2x^2 - 12x + 6 \\
 & \quad\; -2x^2 \quad\quad\; - 8 \\
\hline
 & \quad\quad\quad -12x + 14
\end{array}
$$

Hence $\dfrac{3x^3 - 2x^2 + 6}{x^2 + 4} = 3x - 2 + \dfrac{-12x + 14}{x^2 + 4}$.

YOU CAN NOW ATTEMPT THE WORKSHEET "ALGEBRAIC LONG DIVISION".

PARTIAL FRACTIONS FOR IMPROPER RATIONAL FUNCTIONS

An improper rational function can be expressed as the sum of a polynomial and partial fractions.

Worked Example

Express $\frac{x^3 - 3x}{x^2 - x - 2}$ as the sum of a polynomial and partial fractions.

Solution

$$\begin{array}{r|l} & \quad x + 1 \\ \hline x^2 - x - 2 & x^3 + 0x^2 - 3x + 0 \\ & x^3 - x^2 - 2x \\ \hline & \quad\quad x^2 - x + 0 \\ & \quad\quad x^2 - x - 2 \\ \hline & \quad\quad\quad\quad 2 \end{array}$$

Hence $\frac{x^3 - 3x}{x^2 - x - 2} = x + 1 + \frac{2}{x^2 - x - 2}$ ...(*)

Now express $\frac{2}{x^2 - x - 2}$ in partial fractions.

Note that $\frac{2}{x^2 - x - 2} = \frac{2}{(x-2)(x+1)}$ and that the denominator **contains distinct linear factors**.

$$\frac{2}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$
$$= \frac{A(x+1) + B(x-2)}{(x-2)(x+1)}$$

Hence $2 = A(x+1) + B(x-2)$.

Put $x = -1$ $\Rightarrow$ $2 = A(0) + B(-3)$
$\Rightarrow$ $-3B = 2$
$\Rightarrow$ $B = -\frac{2}{3}$

Put $x = 2$ $\Rightarrow$ $2 = A(3) + B(0)$
$\Rightarrow$ $3A = 2$
$\Rightarrow$ $A = \frac{2}{3}$

Hence $\frac{2}{x^2 - x - 2} = \frac{\frac{2}{3}}{x - 2} - \frac{\frac{2}{3}}{x+1} = \frac{2}{3(x-2)} - \frac{2}{3(x+1)}$.

From (*): $\frac{x^3 - 3x}{x^2 - x - 2} = x + 1 + \frac{2}{3(x-2)} - \frac{2}{3(x+1)}$.

YOU CAN NOW ATTEMPT THE WORKSHEET "PARTIAL FRACTIONS FOR IMPROPER RATIONAL FUNCTIONS."