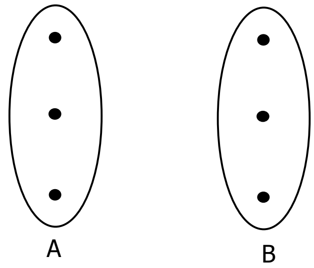
Inverse Functions

If a function links every number in the domain to **only** one number in the range, the function is called a **one to one correspondence**.

When function f(x) is a one to one correspondence from A to B, the function which maps from B back to A is called the **inverse function**, written **$f^{-1}(x)$.**

For example, if f(x) = 2x, the inverse would be the function which "cancels out" multiplication by 2, i.e. $f^{-1}(x) = \frac{1}{2}x$



Finding the Formula of an Inverse Function

We can find the formula for the inverse of a function through a process very similar to changing the subject of a formula.

Example 1: For each function shown find a formula for the inverse function.

a) $f(x) = 2x + 5$

b) $g(x) = \frac{1}{2}(x - 9)$

c) $p(x) = 3x^3 - 4$

d) $h(x) = \frac{3x + 17}{x - 4}$

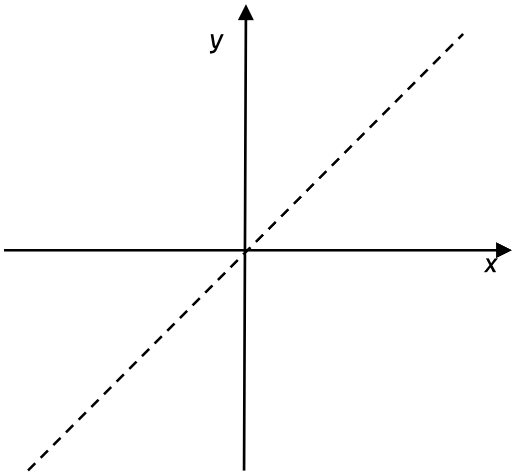
If $f(g(x)) = x$, then $f(x)$ and $g(x)$ are inverse functions, so that

$f(x) = g^{-1}(x)$ AND $g(x) = f^{-1}(x)$

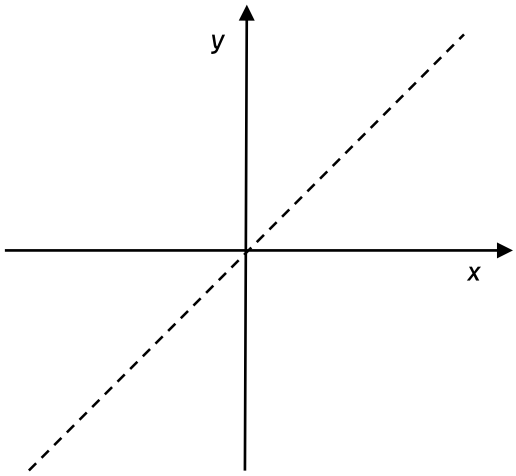
Example 2: $f(x) = 2x + 5$ and $g(x) = \frac{x - 5}{2}$. Show that $g(x) = f^{-1}(x)$

Example 3: Sketch on the same graphs below:

a) $y = f(x)$ and $y = f^{-1}(x)$ where $f(x) = 2x + 5$



b) $y = f(x)$ and $y = f^{-1}(x)$ where $f(x) = \frac{x}{2} - 9$

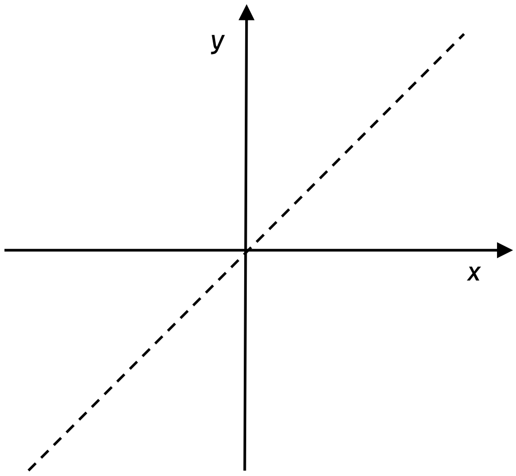


The dotted lines on each diagram are the line y = x. In each case, the graph of an inverse function can be obtained from the graph of the original function by **reflecting in the line y = x.**

Example 4: $g(x) = x^3 + 6$

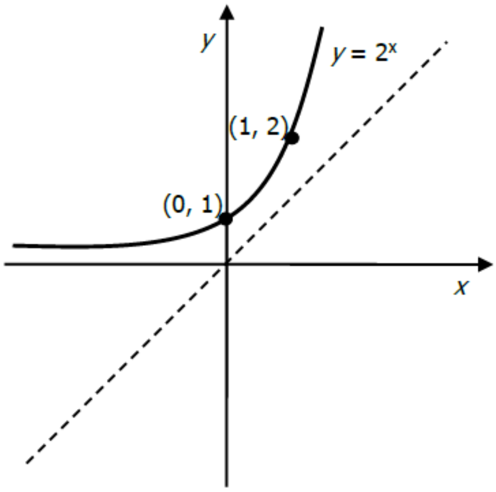
a) Sketch the graph of $y = g(x)$.

b) Show that $g^{-1}(x) = \sqrt[3]{x-6}$



c) Hence sketch the graph of $y = \sqrt[3]{x-6}$

Exponential functions have the formula $f(x) = a^x$, $x \in \mathbf{R}$, where a is called the **base.**
The graph of $y = 2^x$ is shown below.



The graph of $y = 2^x$ passes through the points (0, 1) and (1, 2). As reflection in the line y = x will produce the inverse of $y = 2^x$, then the inverse of $f(x) = 2^x$ must pass through the points (1, 0) and (2, 1)

The inverse of an exponential function is known as a **logarithmic** function.

If $f(x) = a^x$, then $f^{-1}(x) = \log_a x$
(" log to the base a of x")

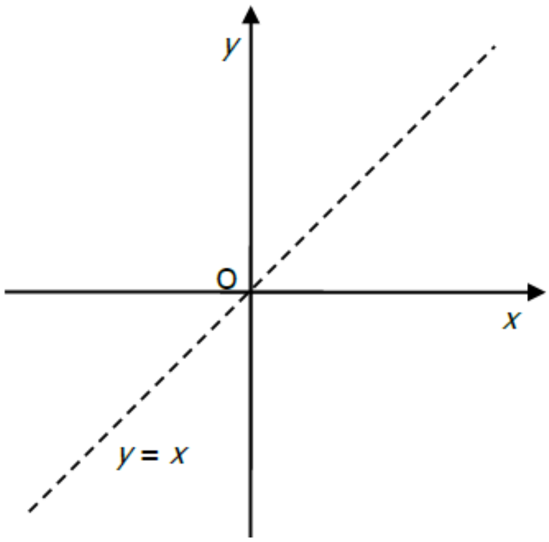
Example 5: Add the graph of $y = \log_2 x$ to the graph opposite.

Note that:

$y = a^x$ passes through (0, 1) and (1, a)
$y = \log_a x$ passes through (1, 0) and (a, 1)

For logarithms:

If $y = a^x$
then
$\log_a y = x$



Example 6: On the graph above, sketch and annotate the graphs of:

a) $y = 5^x$

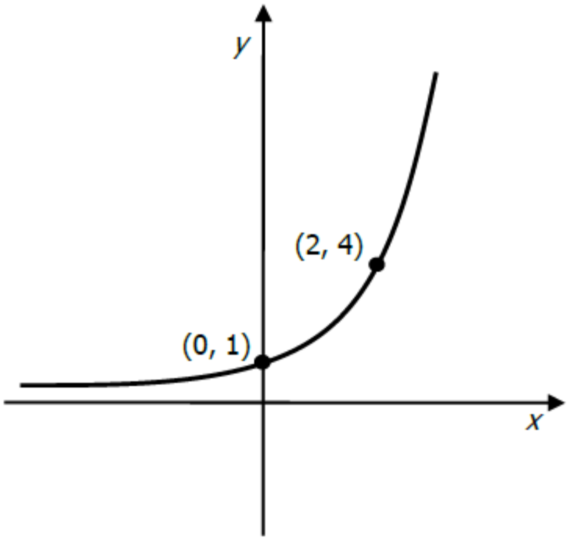
b) $y = \log_5 x$

Example 7: Write as logarithms:

a) $y = 3^x$

b) $q = 13^r$

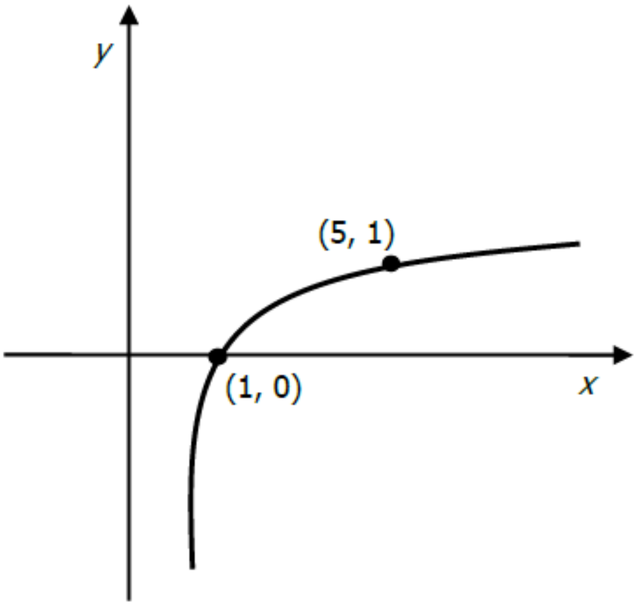
$y = a^x$ means " a multiplied by itself ***x*** times gives y"
$y = \log_a x$ means " y is the number of times I multiply ***a* by itself to get *x***"



Example 8: Shown is the graph of the function $f(x) = 2^x$. To the diagram opposite, add the annotated graphs of the functions:

a) $y = 2^x - 3$

b) $y = 2^{(x - 2)}$



Example 9: Shown is the graph of the function $y = \log_5 x$. To the diagram opposite, add the annotated graphs of the functions:

a) $y = 2\log_5 x$

b) $y = \log_5(x + 1)$

Past Paper Example:

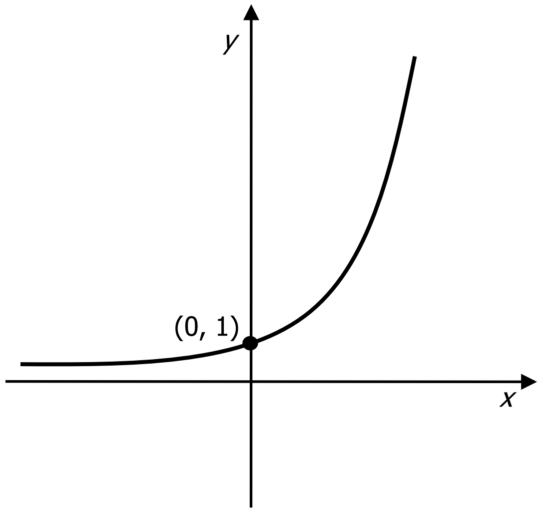
Functions f and g are defined on the set of real numbers. The inverse functions f^{-1} and g^{-1} both exist.

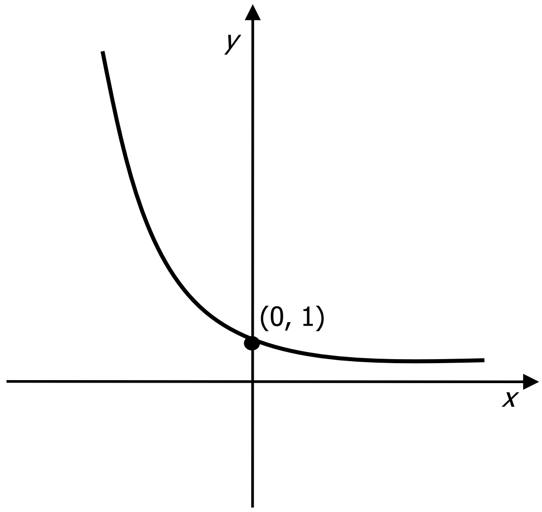
a) Given $f(x) = 3x + 5$, find $f^{-1}(x)$

b) If $g(2) = 7$, write down the value of $g^{-1}(7)$.

Exponential and Logarithmic Functions

Exponential functions are those with variable powers, e.g. $y = a^x$. Their graphs take two forms:





When a > 1, the graph:

- is always increasing
- is always positive
- never cuts the *x* – axis
- passes through (0, 1)
- shows **exponential growth**

When 0 < a < 1, the graph

- is always decreasing
- is always positive
- never cuts the *x* – axis
- passes through (0, 1)
- shows **exponential decay**

Exponential Functions as Models

Example 11: Ulanda's population in 2016 was 100 million and it was growing at 6% per annum.

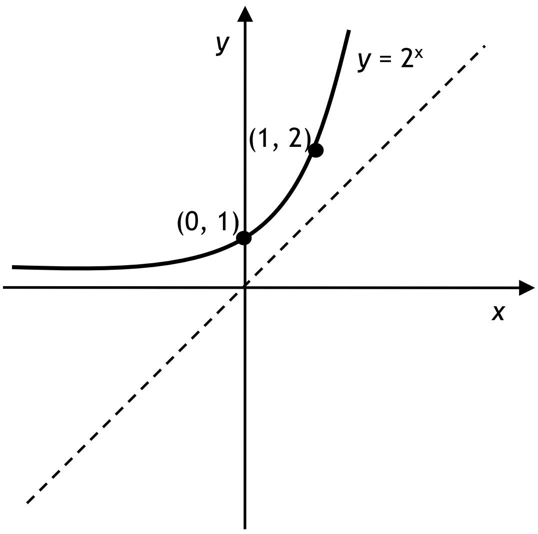
a) Find a formula P_n for the population in millions, n years later.

b) Estimate the population in the year 2026

Example 12: 8000 gallons of oil are lost in an oil spill in Blue Sky Bay. At the beginning of each week a filter plant removes 67% of the oil present.

a) Find a formula G_n for the amount of oil left in the bay after n weeks.

b) After how many **complete** weeks will there be less than 10 gallons left?



The inverse of an exponential function is known as a **logarithmic** function.

If $f(x) = a^x$, then $f^{-1}(x) = \log_a x$

("log to the base a of x")

We have seen that the graph of the inverse of a function can be obtained by reflection in the line $y = x$.

Since the graph of $y = 2^x$ passes through the points (0, 1) and (1, 2), then the inverse of $f(x) = 2^x$ must pass through the points (1, 0) and (2, 1).

Example 13: Add the graph of $y = \log_2 x$ to the graph opposite.

Note that:

$y = a^x$ passes through (0, 1) and (1, a)

$y = \log_a x$ passes through (1, 0) and (a, 1)

$y = a^x$ means "***a* multiplied by itself *x* times gives *y***"

$y = \log_a x$ means "***y* is the number of times I multiply *a* by itself to get *x***"

Since the graph does not cross the *y*-axis, we can only take the logarithm of a positive number

The expression "$\log_a x$" can be read as "*a* to the power of what is equal to *x* ?", e.g. $\log_2 8$ means "2 to the power of what equals 8?", so $\log_2 8 = 3$.

Example 14: Write in logarithmic form:

a) $5^2 = 25$

b) $12^1 = 12$

c) $8^{1/3} = 2$

d) $8^x = y$

e) $1 = q^0$

f) $(x - 3)^4 = k$

Example 15: Write in exponential form:

a) $3 = \log_5 125$

b) $\log_7 49 = 2$

c) $\log_4 4096 = 6$

d) $\log_2\left(\frac{1}{4}\right) = -2$

e) $\log_b g = 5h$

f) $1 = \log_7 7$

Example 16: Evaluate:

a) $\log_8 64$

b) $\log_2 32$

c) $\log_{3.5} 3.5$

d) $\log_{25} 5$

e) $\log_4\left(\frac{1}{2}\right)$

Since $a^1 = a$, then **$\log_a a = 1$**

Since $a^0 = 1$, then **$\log_a 1 = 0$.**

Laws of Logarithms

$\log_a xy = \log_a x + \log_a y$	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$	$\log_a x^n = n\log_a x$

Example 17:

a) $\log_2 4 + \log_2 8 - \log_2\frac{1}{2}$

b) $2\log_5 10 - \log_5 4$

c) Simplify $\frac{1}{4}(\log_3 810 - \log_3 10)$

Solving Logarithmic Equations

You **MUST** memorise the laws of logarithms to solve log equations! As we can only take logs of **positive** numbers, we must remember to discard any answers which violate this rule!

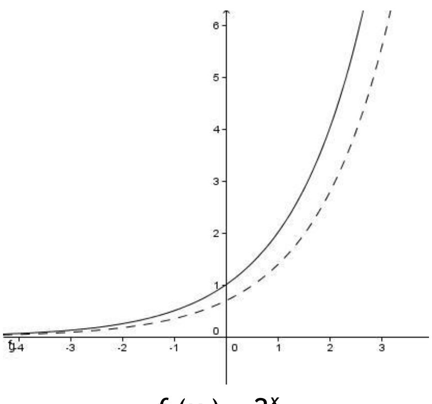
Example 18: Solve:

a) $\log_4(3x - 2) - \log_4(x + 1) = \frac{1}{2}$ $\left(x > \frac{2}{3}\right)$

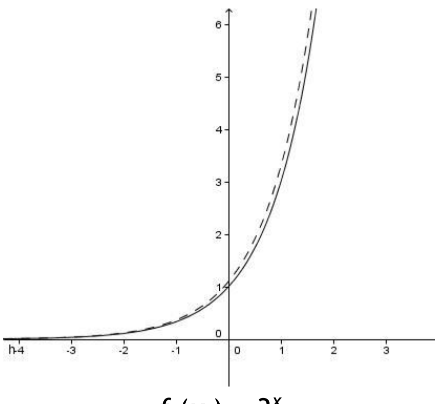
b) $\log_6 x + \log_6(2x - 1) = 2$ $\left(x > \frac{1}{2}\right)$

The Exponential Function and Natural Logarithms

The graph of the derived function of $y = a^x$ can be plotted and compared with the original function. The new graphs are also exponential functions. Below are the graphs of $y = 2^x$ and $y = 3^x$ (solid lines) and their derived functions (dotted).



$f(x) = 2^x$



$f(x) = 3^x$

The derived function of $y = 2^x$ lies **under** the original graph, but the derived function of $y = 3^x$ lies **above** it.

This means that there must be a value of a between 2 and 3 where the derived function lies **on** the original.

i.e. where $f(x) = f'(x)$

The value of the base of this function is known as e, and is approximately 2.71828.

The function $y = e^x$ is known as The Exponential Function.

The function $y = \log_e x$ is known as the Natural Logarithm of x, and is also written as $\ln x$.

Example 19: Evaluate:

a) e^3

b) $\log_e 120$

Example 20: Solve:

a) $\ln x = 5$

b) $5^{x-1} = 16$

Example 21: Atmospheric pressure P_t at various heights above sea level can be determined by using the formula $P_t = P_0 e^{rt}$, where P_0 is the pressure at sea level, t is the height above sea level in thousands of feet, and r is a constant.

a) At 20 000 feet, the air pressure is half that at sea level. Find r accurate to 3 significant figures.

b) Find the height at which P is 10% of that at sea level.

Example 22: A radioactive element decays according to the law $A_t = A_0 e^{kt}$, where A_t is the number of radioactive nuclei present at time t years and A_0 is the initial amount of radioactive nuclei.

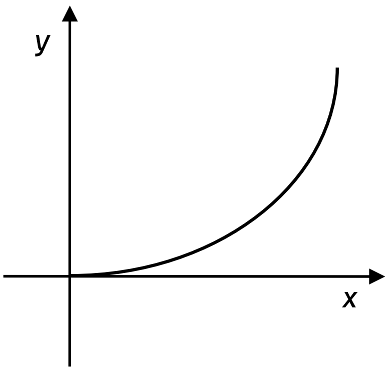
a) After 150 years, 240g of this material had decayed to 200g.
Find the value of k accurate to 3 s.f.

b) The half-life of the element is the time taken half the mass to decay. Find the half-life of the material.

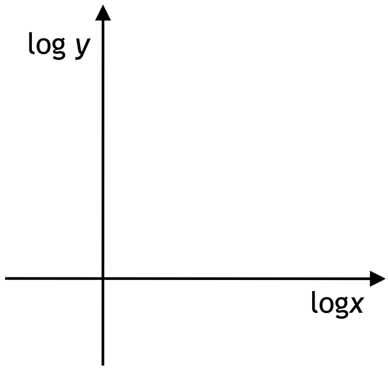
Using Logs to Analyse Data, Type 1: $y = kx^n \Leftrightarrow \log y = n \log x + \log k$

When the data obtained from an experiment results in an exponential graph of the form $y = kx^n$ as shown below, we can use the laws of logarithms to find the values of k and n.

To begin, take logs of both sides of the exponential equation.

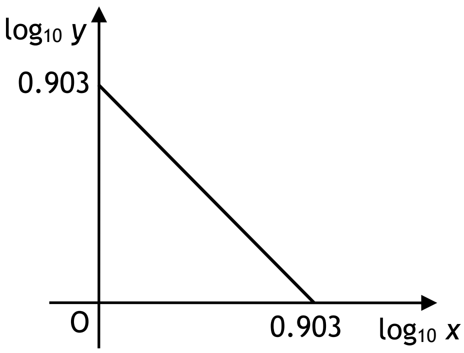


$y = kx^n$



$y = kx^n$ **This gives a straight line graph!** $\log y = n \log x + \log k$

Note: the base is not important, as long as the **same** base is used on both sides.



Example 23: Data are recorded from an experiment and the graph opposite is produced.

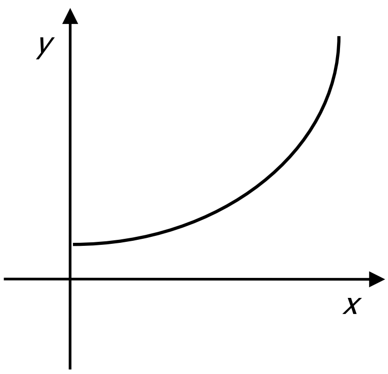
a) Find the equation of the line in terms of $\log_{10} x$ and $\log_{10} y$.

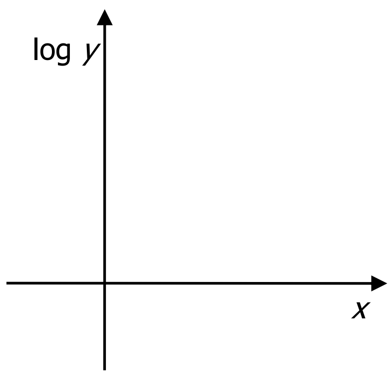
b) Hence express y in terms of x.

Using Logs to Analyse Data, Type 2: $y = kn^x \Leftrightarrow \log y = \log n\,(x) + \log k$

A similar technique can be used when the graph is of the form $y = kn^x$ (i.e. x is the index, not the base as before).

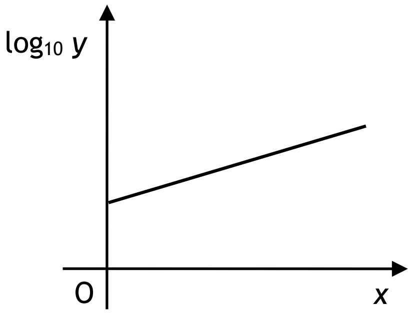
$y = kn^x$





$y = kn^x$ $\log y = (\log n)\, x + \log k$

Example 24: The data below are plotted and the graph shown is obtained.

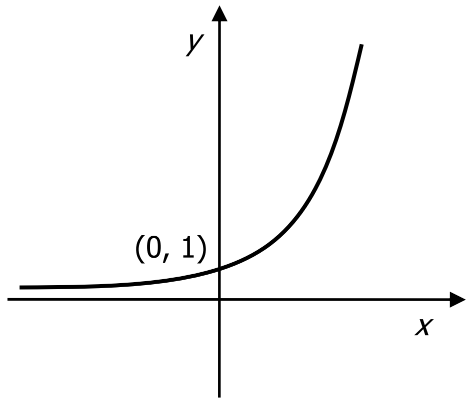


x	0	1	2	3	4
$\log_{10} y$	0.602	1.079	1.556	2.033	2.510

a) Express $\log_{10} y$ in terms of x.

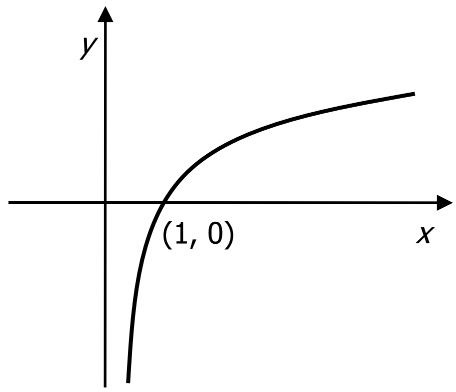
b) Hence express y in terms of x.

$y = e^x + a$



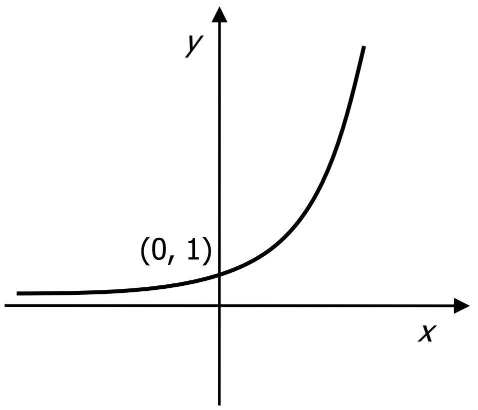
$y = e^x + a$ is obtained by **sliding** $y = e^x$:
Vertically upwards if a>0
Vertically downwards if a<0

$y = \ln(x + a)$



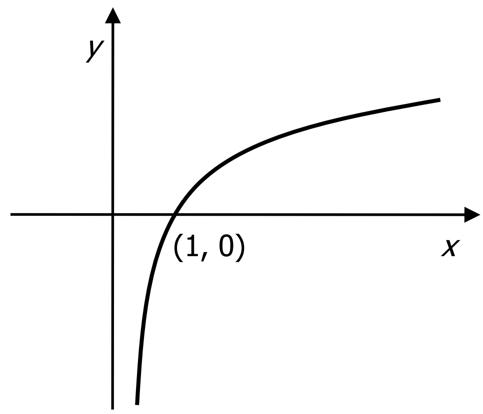
$y = \ln(x + a)$ is obtained by **sliding** $y = \ln x$
Horizontally left if a>0
Horizontally right if a<0

$y = e^{(x + a)}$



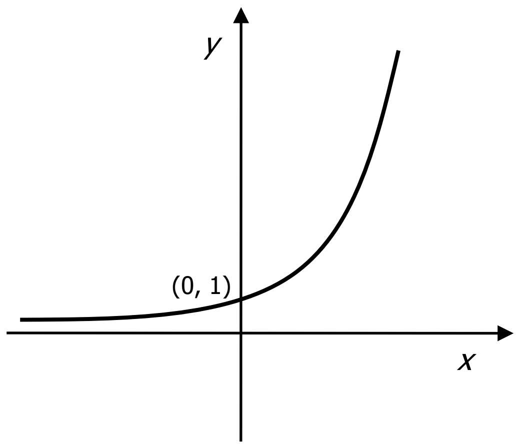
$y = e^{(x + a)}$ is obtained by **sliding** $y = e^x$
Horizontally left if a>0
Horizontally right if a<0

$y = k \ln x$



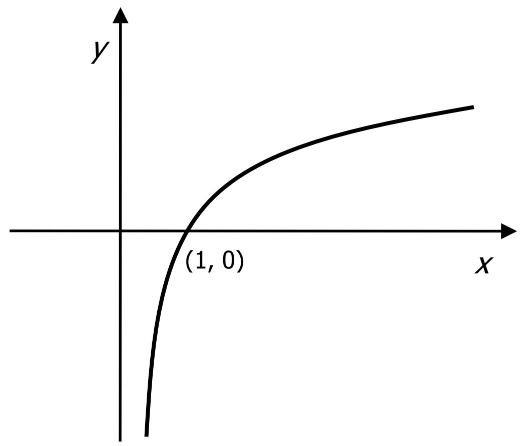
$y = k \ln x$ is obtained by **vertically:**
stretching $y = \ln x$ if $k > 1$
compressing $y = \ln x$ if $0 < k < 1$

$y = e^{-x}$

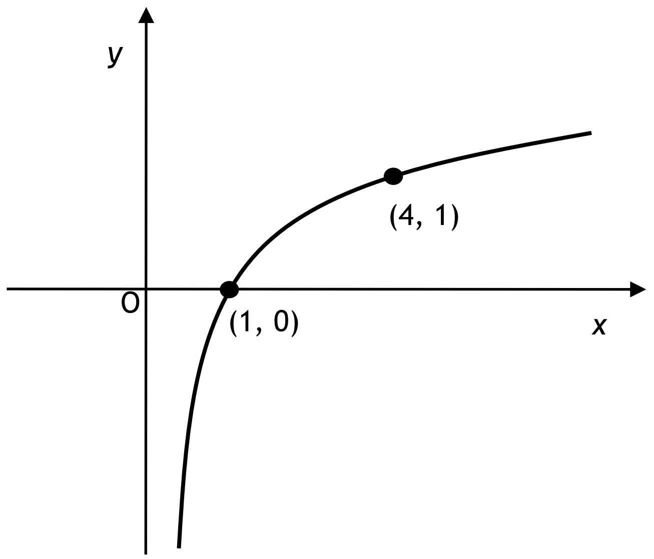


$y = e^{-x}$ is obtained by **reflecting** $y = e^x$**:**
in the y -axis

$y = - \ln x$



$y = - \ln x$ is obtained by **reflecting** $y = \ln x$:
in the x - axis



Example 25: The graph of $y = \log_4 x$ is shown. On the same diagram, sketch:

a) $y = \log_4 4x$

b) $y = \log_4 \left(\frac{1}{4x}\right)$

Past Paper Example 1:

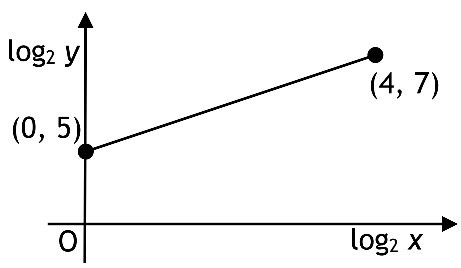
a) Show that $x = 1$ is a root of $x^3 + 8x^2 + 11x - 20 = 0$, and hence factorise $x^3 + 8x^2 + 11x - 20$ fully

b) Solve $\log_2(x + 3) + \log_2(x^2 + 5x - 4) = 3$

Past Paper Example 2: Variables x and y are related by the equation $y = kx^n$.

The graph of $\log_2 y$ against $\log_2 x$ is a straight line through the points (0, 5) and (4, 7), as shown in the diagram.

Find the values of k and n.



Past Paper Example 3: The concentration of the pesticide *Xpesto* in soil is modelled by the equation:

$P_t = P_0 e^{-kt}$ where: P_0 is the initial concentration
P_t is the concentration at time t
t is the time, in days, after the application of the pesticide.

a) Once in the soil, the half-life of a pesticide is the time taken for its concentration to be reduced to one half of its initial value.

If the half-life of *Xpesto* is 25 days, find the value of k to 2 significant figures.

b) Eighty days after the initial application, what is the percentage decrease in *Xpesto*?

Past Paper Example 4: Simplify the expression $3\log_e 2e - 2\log_e 3e$ giving your answer in the form $A + \log_e B - \log_e C$, where A, B and C are whole numbers.

Past Paper Example 5: Two variables x and y satisfy the equation $y = 3\left(4^x\right)$.

A graph is drawn of $\log_{10} y$ against x. Show that its equation will be of the form $\log_{10} y = Px + Q$, and state the gradient and y-intercept of this line.

Expressions & Functions Unit Topic Checklist: Unit Assessment Topics in Bold			
Topic		Questions	Done?
Logs & Exponentials	**Logarithms**	Exercise 15E, (all)	Y/N
	Laws of logarithms	Exercise 15F, Q 1	Y/N
	Log equations	Exercise 15G, Q 1, 2, 3	Y/N
	e^x and Natural logarithms	Exercise 15D, Q 1 – 5	Y/N
	Exponential growth/decay	Exercise 15H, Q 4 - 7	Y/N
	Data and straight line graphs	Exercise 15J, Q 2; Exercise 15K, Q 2	Y/N
	Related log & exponential graphs	Exercise 15K, Q 1 - 7	Y/N
Trigonometry	**Radians**	Exercise 4C, Q 1 – 3	Y/N
	Exact Values	Exercise 4E, Q 1, 3	Y/N
	Trig Identities	Exercise 17, Q 2; Exercise 11J, Q 20	Y/N
	Compound and double angle formulae	Exercise 11D, Q 6 – 8; Exercise 11F, Q 1 – 4, 7, 9	Y/N
	$k\cos(x - \alpha)$	Exercise 16C, Q 1 – 5; Exercise 16D, Q 2	Y/N
	$k\cos(x + \alpha)$	Exercise 16E, Q 1	Y/N
	$k\sin(x \pm \alpha)$	Exercise 16E, Q 2, 3; Exercise 16E, Q 4, 5	Y/N
	Wave Fn Maxima and minima	Exercise 16G, Q 1, 3, 4, 5, 7	Y/N
	Solving Wave Fn equations	Exercise 16H, Q 1 - 4	Y/N
Sets, Functions & Graphs	**Transforming graphs**	Exercise 3P, Q 1 – 9	Y/N
	Naming/Sketching trig graphs	Exercise 4B, (all)	Y/N
	Completing the square	Exercise 8D, Q 4, 6; Exercise 5, Q 3, 4	Y/N
	Graphs of derived functions	Exercise 6P, (all)	Y/N
	Set Notation	Exercise 2A, Q 2 & 3	Y/N
	Composite Functions	Exercise 2C, Q 5 - 10	Y/N
	Inverse Functions	Exercise 2D, Q 2; Exercise 2I, Q 1	Y/N
	Graphs of inverse functions	Exercise 2F, Q 1 & 2	Y/N
	Exponential & log graphs	Exercise 3N, Q 3, 4; Exercise 3O, p 47, Q 2, 3	Y/N
Vectors	**Resultant vectors**	Exercise 13N (all)	Y/N
	Unit Vectors (inc. i, j, k)	Exercise 13F, Q 1, 2;	Y/N
	Collinearity	Exercise 13N, Q 15 – 18, 23	Y/N
	Section Formula	Exercise 13N, Q 20 - 24	Y/N
	Scalar Product	Exercise 13O, Q 1; Exercise 13P, Q 1, 2	Y/N
	Angle between vectors	Exercise 13Q, Q 1, 2; Exercise 13S, Q 4 – 7	Y/N
	Perpendicular Vectors	Exercise 13R, Q 1 - 8	Y/N
	Properties of Scalar Product	Exercise 13U, Q 1, 2, 4, 5	Y/N

Past Paper Questions by Topic

	Topic	2008		2009		2010		2011		2012		2013		2014		2015		2016	
		P1	P2	P1	P2	P1	P2	P1	P2	P1	P2	P1	P2	P1	P2	P1	P2	P1	P2
Functions and Graphs	Ranges and Domains	4b						20				13		12			2c	15b	
	Composite Functions	4a		3		3			2a,b		1a	1			3a	5b	2a	12a	
	Inverse Functions															5a		6	
	Transforming graphs				7			3			4	11		11		13c			
	Interpreting trig functions and graphs	10b				11b	4a			9		4				4			
	Exact Values									1				13					
Recurrence Relation	Terms of Recurrence Relations			4		7a							1	1			3a	3a	
	Creating & using RR formulae																		
	Finding a limit of an RR	6				7b			3c		6	8		10			3b	3b, c	
	Solving RR's to find *a* and *b*								3a,b										
Straight Line	Gradients from points or equations							2				2							
	Gradients from angles with x - axis	1						8		4					1c				
	Equations of straight lines				1	1		21a		23b		5	2a					1	1b
	Perpendicular bisectors		3					21c		23a					1a				
	Altitudes			1b													1a		
	Medians			1a													1b		1a
	Points of intersection of lines		3c	1c				21b		23c			2b,c		1b		1c		1c
	Distance Formula	2b								23d									
	Collinearity															9			
Differentiation	Finding derivatives of functions									6,8						7			
	Equations of tangents to curves		6		3a,10b		5a	4		2				2	2	2		2	
	Increasing & decreasing functions													21a				9b	
	Stationary points			5		9b		22b		18				21b				9a	
	Curve Sketching					9c		22ac											
	Closed Intervals	8c								12	3								
	Graphs of derived functions																		
	Optimisation				12		6						7				8		7
	Velocity, Acceleration, Displacement														9				
Integration	Finding indefinite integrals		1					11,16		11		7							
	Definite Integrals														5				
	Area under a curve			6		8c				21b						12			3b
	Area between two curves		5						4				4		7		4		
	Differential Equations				5		10b									15			9

Past Paper Questions by Topic

	Topic	2008		2009		2010		2011		2012		2013		2014		2015		2016	
		P1	P2	P1	P2	P1	P2	P1	P2	P1	P2	P1	P2	P1	P2	P1	P2	P1	P2
Quadratics	Completing the square			8				5		3		21		17			2b	12b	
	Quadratic Inequations		11b					18		19		19				8			
	Roots using $b^2 - 4ac$		11b		2			9			1b	3			3b				2
	Tangency using $b^2 - 4ac$				3b	4			2c, d			22c,d	3b			11b			
Polynomials	Synthetic Division	8a,b	11a	9b, c		8b		7		21a		6	3a	22		3			3a
	Intersections of curves																		
	Functions from graphs						10a	17		13		17		15				15a	
	Approximate roots (Iteration)																		
Circles	$(x - a)^2 + (y - b)^2 = r^2$	11a	3c						7		2b			23c				4	4a
	General equation of a circle	2a		2a	4	5			7			22a			8				4a
	Lines cutting circles										2a			23ab		14	5		
	Tangents to circles	11b	3b	2b			3, 5c	6				22b		2		11a		8	
	Distance between centres vs radii																		4b
Trigonometry	Trig Equations		8	7		6	4b	10	6b			15	8		6				
	Compound angle formulae		2		8b		2a	12				10						13	8b
	Double angle formulae	9	8	7	8a,b	6	2b	23		5		9		7,18		10			
	Trigonometric identities																7b		11a
Vectors	Interpreting vector diagrams													19	4a				
	Unit Vectors				6					15								11b	
	Position Vectors and Components	3b				2		1	1a, b	10	5ai	12		6	4b			7	5a
	Collinearity							15				24							
	Section Formula	3a								7								11a	
	Scalar Product (angle form)																		
	Scalar Product (components)			9a						17									5b
	Angle between vectors		4	9d			1		1c		5aii,b				4c				5b
	Perpendicular Vectors													14		1			
	Properties of Scalar Product		10					14				14		16			6		
WF	Wave Functions	10a			10a	11a			6a	22a		23		4,9			9		8a
Further Calculus	Chain Rule (inc. trig functions)	5				10		13		16		18		8					10a, 11c
	Integrating $a(\ldots)^n$									14,22b		16		5	5				10b
	Integrating sin and cos	5			9, 10b		7		6b				6				7a,c	5	
Logs and Exponentials	Exponential growth/decay		9		11								9						6
	Log equations		7				8			20			5	3,20		6		14	
	Exponential and log graphs	7					9	19			7					13a, b		10	
	Linearisation			10			11		5			20		24					