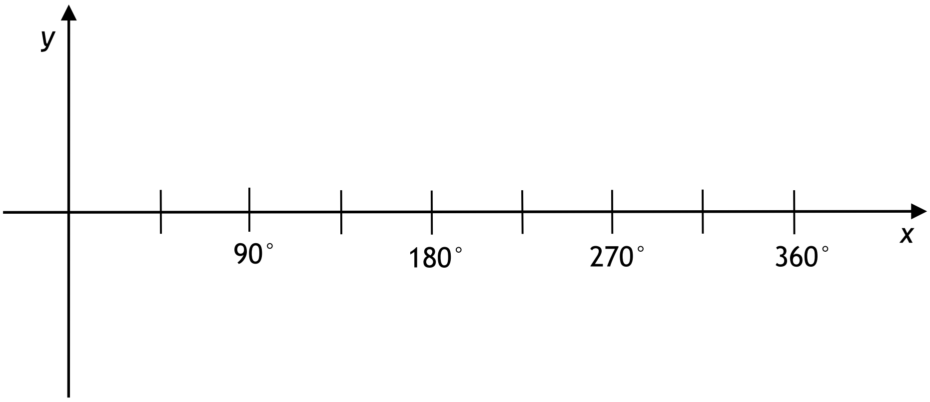
Trigonometry: The Wave Function

It is possible to model the behaviour of waves in real-life situations (e.g. the interaction of sound waves or the tides where two bodies of water meet) using trigonometry. Consider the result of combining the waves represented by the functions $y = \sin x°$ and $y = \cos x°$. To find what the resultant graph would look like, complete the table of values (accurate to 1 d.p.) and plot on the axes below.

	0°	45°	90°	135°	180°	225°	270°	315°	360°
$\sin x°$									
$\cos x°$									
$\sin x° + \cos x°$									



Max = ____________

Min = ____________

Max when x = ____________

Min when x = ____________

$\therefore$ **y =** ______________________

Looking at the graph of $y = \sin x° + \cos x°$ above, we can compare it to cosine graph shifted 45° to the right (i.e. $y = \cos(x - \alpha)°$), and stretched vertically by a factor of roughly 1.4 (i.e. $y = k\cos x°$).

It is important to note, however, that the graph could also be described as a cosine graph shifted to the *left*, and also as a sine graph! Therefore, $y = \sin x° + \cos x°$ could **also** be written as:

y = 1.4cos(x + ________) OR **y = 1.4sin(x - ________)** OR **y = 1.4sin(x + ________)**

Rather than drawing an approximate graph, it is more useful if we use an algebraic method.

NOTE: you will only be asked to use one specific form to describe a function, not all four!

Example 1: Write $\sin x° + \cos x°$ in the form $k\cos(x - \alpha)°$, where $0 \le \alpha \le 360$.

This technique can also include the difference between waves and to include double (or higher) angles, **but only when the angles of both the sin and cos term are the same (**i.e. $2\cos 2x + 5\sin 2x$ can be written as a wave function, but $2\cos 2x + 5\sin 3x$ could not).

Example 2: Write $\sin x - \sqrt{3}\cos x$ in the form $k\cos(x - \alpha)$, where $0 \le \alpha \le 2\pi$

Example 3: Write $12\cos x° - 5\sin x°$ in the form $k\sin(x - \alpha)°$, where $0 \le \alpha \le 360$

Example 4: Write $2\sin 2\theta - \cos 2\theta$ in the form $k\sin(2\theta + \alpha)$, where $0 \le \alpha \le 2\pi$

Solving Trig Equations Using the Wave Function

In **almost all** cases, questions like these will be split into two parts, with a) being a "write ………. in the form $y = k\cos(x - \alpha)$" followed by b) asking "hence or otherwise solve………..".

Use the wave function from part a) to solve the equation!

Example 5:

a) Write $2\cos x° - \sin x°$ in the form $k\cos(x - \alpha)°$ where $0 \le \alpha \le 360$

b) Hence solve $2\cos x° - \sin x° = -1$ where $0 \le x \le 360$

Maximum and Minimum Values and Sketching Wave Function Graphs

Look back at the graph you drew of $\sin x° + \cos x°$. The maximum value of the graph is $\sqrt{2}$ at the point where $x = 45°$, and the minimum value is $-\sqrt{2}$ at the point where $x = 225°$. Compare these to the maximum and minimum of $y = \cos x°$, i.e. a maximum of 1 where $x = 0°$ or $360°$ and a minimum of -1 where $x = 180°$.

Since $\sin x° + \cos x° = \sqrt{2}\cos(x - 45)°$, we can see that the maximum and minimum values change from ± 1 to $\pm k$.

The maximum value occurs where $\sqrt{2}\cos(x - 45)° = \sqrt{2}$, i.e. $\cos(x - 45)° = 1$. Similarly, the minimum value occurs where $\sqrt{2}\cos(x - 45)° = -\sqrt{2}$, i.e. $\cos(x - 45)° = -1$

For $a\sin x + b\cos x = k\cos(x - \alpha)$, $k > 0$:

Maximum = k
when $\cos(x - \alpha) = 1$

Minimum = $-k$
when $\cos(x - \alpha) = -1$

Example 6:

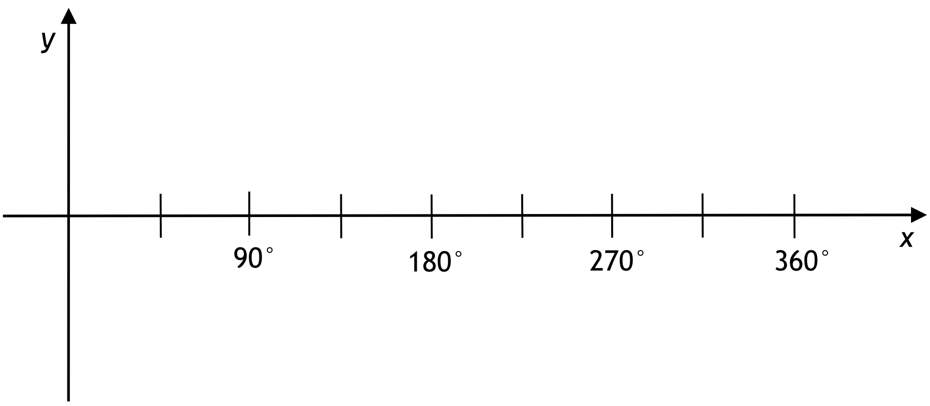
a) Write $\sqrt{3}\sin x + \cos x$ in the form $k\cos(x - \alpha)^\circ$, where $0 \le \alpha \le 360^\circ$

b) Find algebraically for $0 \le x \le 360^\circ$:

(i) The maximum and minimum turning points of $y = \sqrt{3}\sin x^\circ + \cos x^\circ$.

(ii) The points of intersection of $y = \sqrt{3}\sin x^\circ + \cos x^\circ$ with the coordinate axes.

c) Sketch and annotate the graph of $y = \sqrt{3}\sin x^\circ + \cos x^\circ$ for $0 \le x \le 360^\circ$.



Recognising Trig Equations

The trig equations we can be asked to solve at Higher level can be split into three types based on the **angle** (i.e. $x°$, $2x°$, $3x°$ etc) and the **function(s)** (i.e. sin, cos, tan, sin & cos).

Type	e.g.:	Method
Type One: **One Function** **One Angle**	$2\sin 4x + 1 = 0$ $\tan^2 x = 3$ $3\sin^2 x - 4\sin x + 1 = 0$	**1. Factorise (if necessary)** **2.** Rearrange to sin(..) = (..) [or cos, or tan] **3. Inverse sin/cos/tan to solve**
Type Two: **Two Functions** **One Angle**	$\sin x + \cos x = 1$ $3\cos(2x) + 4\sin(2x) = 0$ $\cos(4\theta) - \sqrt{3}\sin(4\theta) = -1$	**1. Rewrite as a WAVE FUNCTION** **(choose $k\cos(x - \alpha)$ unless told differently)** **2. Solve as Type One**
Type Three: **Two Angles**	$5\cos(2\theta) = \cos\theta - 2$ $2\sin(2x) + \sin(x) = -0.5$ $2\cos 2x - \sin x + 5 = 0$	**1. Rewrite the double angle and factorise** **(change cos2x to the SINGLE ANGLE function)** **2. Solve as Type One**

Past Paper Example:

a) The expression $\sqrt{3}\sin x° - \cos x°$ can be written in the form $k\sin(x - \alpha)°$, where $k > 0$ and $0 \leq \alpha < 360$.

Calculate the values of k and α.

b) Determine the maximum value of $4 + 5\cos x° - 5\sqrt{3}\sin x°$, where $0 \leq \alpha < 360$, and state the value of x for which it occurs.