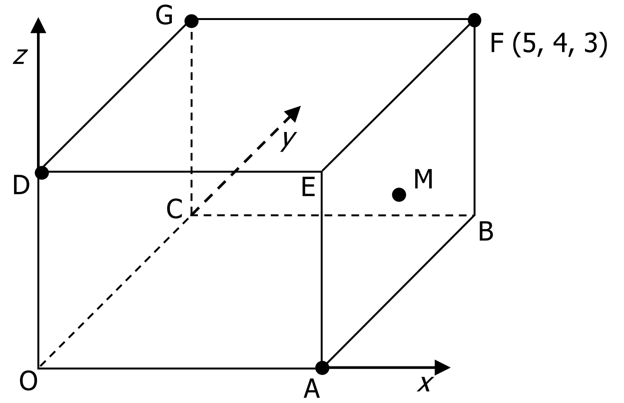
Vectors

Revision from National 5

A measurement which only describes the magnitude (i.e. size) of the object is called a **scalar** quantity, e.g. Glasgow is 11 miles from Airdrie. A **vector** is a quantity with **magnitude and direction**, e.g. Glasgow is 11 miles from Airdrie on a bearing of 270°.

The position of a point in 3-D space can be described if we add a third coordinate to indicate height.



Example 1: OABC DEFG is a cuboid, where F is the point (5, 4, 3). Write down the coordinates of the points:

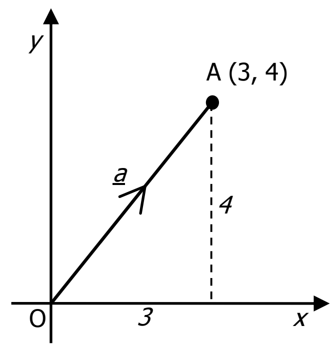
a) A

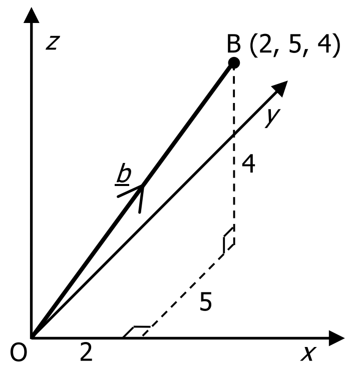
b) D

c) G

d) M, the centre of face ABFE

The rules of vectors can be used in either 2 or 3 dimensions:





Directed line segment $\overrightarrow{OA}$	Directed line segment $\overrightarrow{OB}$
Position vector $\underline{a}$	**Position vector $\underline{b}$**
Components $\begin{pmatrix}3\\4\end{pmatrix}$	**Components** $\begin{pmatrix}2\\5\\4\end{pmatrix}$

The **magnitude** of a vector is its length, which can be determined by Pythagoras' Theorem. The magnitude of $\underline{a}$ is written as $|\underline{a}|$.

If $\underline{u} = \begin{pmatrix}a\\b\end{pmatrix}$, then $|\underline{u}| = \sqrt{a^2 + b^2}$

If $\underline{u} = \begin{pmatrix}a\\b\\c\end{pmatrix}$, then $|\underline{u}| = \sqrt{a^2 + b^2 + c^2}$

Example 2: Determine $|\underline{a}|$ and $|\underline{b}|$ in the examples above.

Addition of Vectors

Two (or more) vectors can be added together to produce a **resultant vector.**

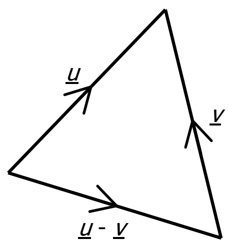
In general: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ and **If $\underline{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$, then $\underline{u} + \underline{v} = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix}$**

Example 3: Find $\underline{p} + \underline{q}$ when $\underline{p} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\underline{q} = \begin{pmatrix} 7 \\ -7 \\ 4 \end{pmatrix}$.

Example 4: Find values of x and y such that

$$\begin{pmatrix} x \\ 4 \end{pmatrix} + \begin{pmatrix} 12 \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \end{pmatrix}$$

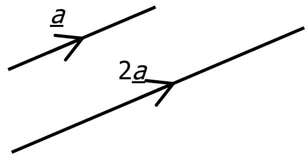
Subtraction of Vectors



Subtraction of vectors can be considered as going along the second vector in the **wrong direction**.

If $\underline{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$, then $\underline{u} - \underline{v} = \begin{pmatrix} a-d \\ b-e \\ c-f \end{pmatrix}$

Multiplication by a Scalar Quantity



If we go along $\underline{a}$ twice, the resultant vector is $\underline{a} + \underline{a} = 2\underline{a}$. As we have not changed direction, it follows that $2\underline{a}$ must be **parallel** to $\underline{a}$.

If $\underline{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then $k\underline{u} = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}$

If $\underline{v} = k\underline{u}$, then $\underline{u}$ and $\underline{v}$ are parallel

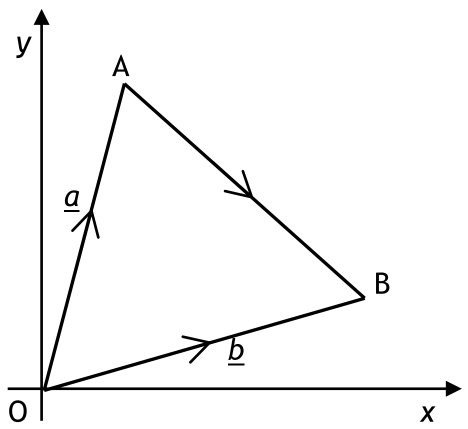
Example 5: If $\underline{b} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}$ and $\underline{c} = \begin{pmatrix} -3 \\ -5 \\ 5 \end{pmatrix}$, find:

a) $3\underline{b}$

b) $2\underline{b} + \underline{c}$

c) $\underline{c} - \frac{1}{2}\underline{b}$

Position Vectors



Consider the vector $\overrightarrow{AB}$ in the diagram opposite. $\overrightarrow{AB}$ is the resultant vector of going along $\underline{a}$ in the opposite direction, followed by $\underline{b}$ in the correct direction.

So, $\overrightarrow{AB} = -\underline{a} + \underline{b}$, i.e.:

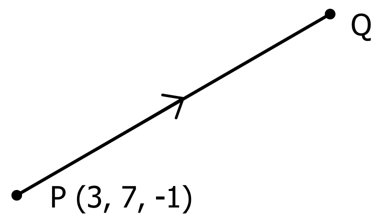
$$\overrightarrow{AB} = \underline{b} - \underline{a}$$

Example 6: L is the point (4, -7, 2), M is the point (-5, -3, -1).

Find the components of $\overrightarrow{LM}$.

Example 7: P is the point (3, 7, -1). $\overrightarrow{PQ}$ has components $\begin{pmatrix} -4 \\ 9 \\ -3 \end{pmatrix}$.

Find the coordinates of Q.



Unit Vectors

A **unit vector** is a vector with magnitude = 1.

Example 8: Find the components of the unit vector parallel to $\underline{h} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

To find the components of a unit vector:

- **Find the magnitude of the given vector**
- **Divide components by the magnitude**

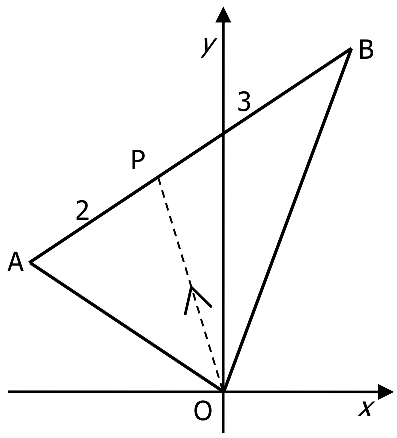
Collinearity

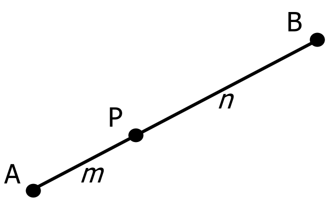
Example 9: Points F, G and H have coordinates (6, 1, 5), G (4, 4, 4), and (-2, 13, 1) respectively. Show that F, G and H are collinear, and find the ratio in which G divides $\overrightarrow{FH}$.

The Section Formula

P divides $\overrightarrow{AB}$ in the ratio 2:3. By examining the diagram, we can find a formula for $\underline{p}$ (i.e. $\overrightarrow{OP}$).

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$





In general, if P divides AB in the ratio m:n, then:

$$\underline{p} = \frac{1}{n+m}(n\underline{a} + m\underline{b})$$

Example 10: A is the point (3, -1, 2) and B is the point (7, -5, 14). Find the coordinates of P such that P divides AB in the ratio 1:3.

Vectors in 3D can also be described in terms of the three unit vectors $\underline{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\underline{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, and $\underline{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$, which are parallel to the x, y, and z axes respectively.

Example 11: $\underline{u} = 3\underline{i} + 2\underline{j} - 6\underline{k}$, $\underline{v} = -\underline{i} + 5\underline{j}$.

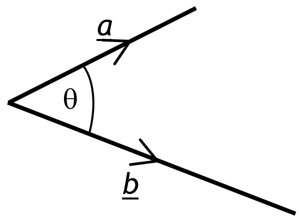
a) Express $\underline{u} + \underline{v}$ in component form

b) Find $|\underline{u} + \underline{v}|$

The Scalar Product (Angle Form)

The **scalar product** is the result of a type of multiplication of two vectors to give a scalar quantity. (i.e. a number with no directional component)

For vectors $\underline{a}$ and $\underline{b}$, the scalar product (or **dot** product) is given as:

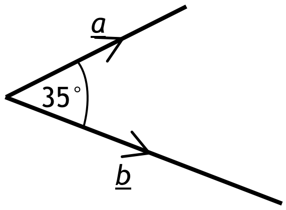


$$\underline{\boldsymbol{a}}.\underline{\boldsymbol{b}} = |\underline{\boldsymbol{a}}||\underline{\boldsymbol{b}}|\cos\theta$$

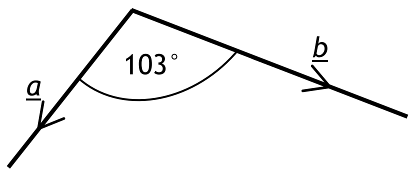
Note:
- $\underline{a}$ and $\underline{b}$ point **away** from the vertex
- $0 \le \theta \le 180^\circ$

Example 12: Find the scalar product in each case below, where $|\underline{a}| = 6$ and $|\underline{b}| = 10$.

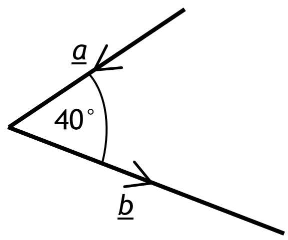
a)



b)



c)



The Scalar Product (Component Form)

We can use the formula below to find the scalar product when we have been given the component forms of the two vectors but not the angle in between them.

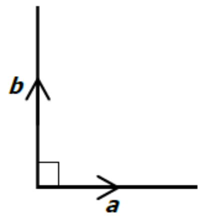
If $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then $\underline{a} \cdot \underline{b} = a_1b_1 + a_2b_2 + a_3b_3$

Example 13: $\underline{a} = \underline{i} + 2\underline{j} + 2\underline{k}$, and $\underline{b} = 2\underline{i} + 3\underline{j} - 6\underline{k}$. Evaluate $\underline{a} \cdot \underline{b}$

Example 14: A is the point (1, 2, 3), B is the point (6, 5, 4) and C is the point (-1, -2, -6).
Evaluate $\overrightarrow{AB}.\overrightarrow{BC}$

Perpendicular Vectors

A special case of the scalar product occurs when we have perpendicular vectors i.e. when $\theta = 90°$:



$$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos 90°$$
$$= |\boldsymbol{a}||\boldsymbol{b}| \times 0$$
$$= 0$$

If $\boldsymbol{a}.\boldsymbol{b} = 0$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular

Example 15: P, Q and R are the points (1, 1, 2), (-1, -1, 0) and (3, -4, -1) respectively. Find the components of $\overrightarrow{QP}$ and $\overrightarrow{QR}$, and hence show that the vectors are perpendicular.

The Angle Between Two Vectors

We can rearrange the angle form of the scalar product to give $\cos\theta = \dfrac{\underline{a}.\underline{b}}{|\underline{a}|\,|\underline{b}|}$.

More specifically: $\cos ABC = \dfrac{\overrightarrow{BA}.\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}$

If the question gives you three points, you MUST find the components of the vectors pointing AWAY from the vertex first!

Example 16: $\underline{a} = \underline{i} + 2\underline{j} + 2\underline{k}$ and $\underline{b} = 2\underline{i} + 3\underline{j} - 6\underline{k}$. Find the angle between $\underline{a}$ and $\underline{b}$.

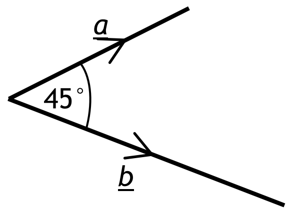
Example 17: A is the point (1, 2, 3), B (6, 5, 4), and C (-1, -2, -6). Calculate $\angle ABC$.

Other Uses of the Scalar Product

For vectors $\underline{a}$, $\underline{b}$, and $\underline{c}$:

$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$

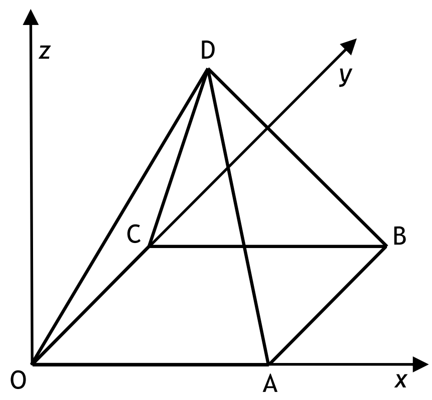
$\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$



Example 18: $|\underline{a}| = 5$ and $|\underline{b}| = 8$. Find $\underline{a} \cdot (\underline{a} + \underline{b})$

Past Paper Example 1: The diagram shows a square-based pyramid of height 8 units. Square OABC has a side length of 6 units. The coordinates of A and D are (6, 0, 0) and (3, 3, 8). C lies on the *y* – axis.

a) Write down the coordinates of B.



b) Determine the components of $\overrightarrow{DA}$ and $\overrightarrow{DB}$.

c) Calculate the size of $\angle$ADB.

Past Paper Example 2:

a) Show that the points A (-7, -8, 1), T (3, 2, 5) and B (18, 17, 11) are collinear and state the ratio in which T divides AB.

b) The point C lies on the *x*-axis.

If TB and TC are perpendicular, find the coordinates of C.

Past Paper Example 3: PQRSTU is a regular hexagon of side 2 units. $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{RS}$ represent the vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ respectively. Find the value of $\underline{a} \bullet (\underline{b} + \underline{c})$