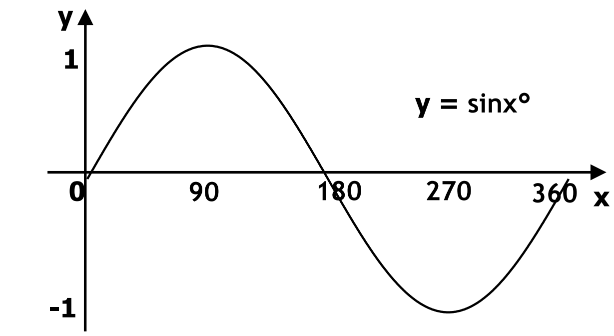
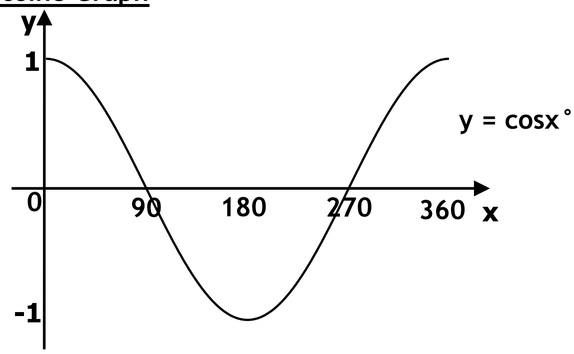
Trigonometry: Addition Formulae and Equations

What you must know from National 5!!!

Sine Graph



Cosine Graph

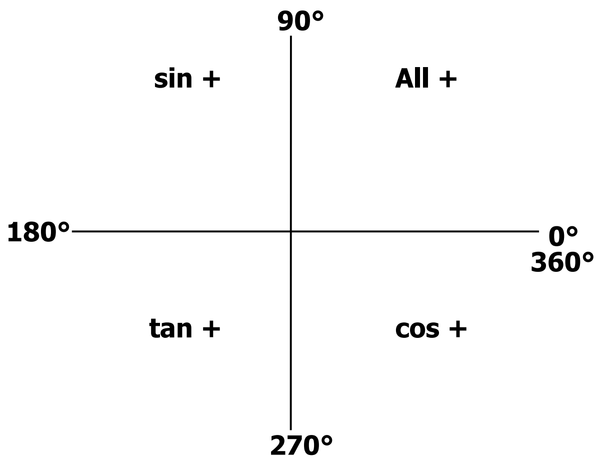


We can use the above graphs to find the values of:

$\sin 0^\circ = 0$	$\cos 0^\circ = 1$
$\sin 90^\circ = 1$	$\cos 90^\circ = 0$
$\sin 180^\circ = 0$	$\cos 180^\circ = -1$
$\sin 270^\circ = -1$	$\cos 270^\circ = 0$
$\sin 360^\circ = 0$	$\cos 360^\circ = 1$

We can use these graphs to solve the following:

$\sin x^\circ = 0$ $(0 \le x \le 360)$ $x = 0^\circ, 180^\circ, 360^\circ$	$\sin x^\circ = -1$ $(0 \le x \le 360)$ $x = 270^\circ$	$\sin x^\circ = 1$ $(0 \le x \le 360)$ $x = 90^\circ$
$\cos x^\circ = 0$ $(0 \le x \le 360)$ $x = 90^\circ, 270^\circ$	$\cos x^\circ = -1$ $(0 \le x \le 360)$ $x = 270^\circ$	$\cos x^\circ = -1$ $(0 \le x \le 360)$ $x = 0^\circ, 360^\circ$



Remember , this means that:

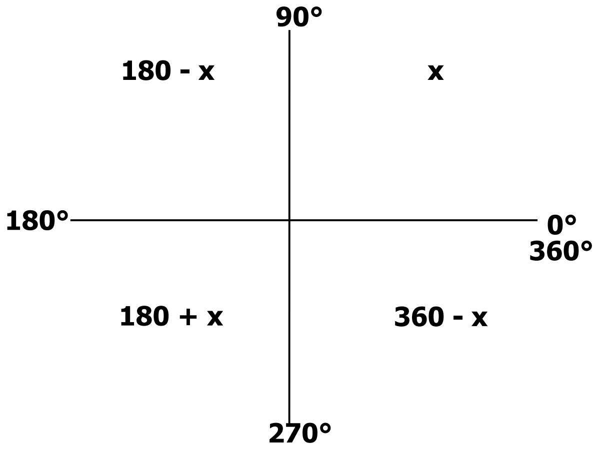
sin 160° would be +

cos 200° would be -

tan 200° would be +

sin 320° would be -
and so on...

Related Angles



This diagram can be used to find families of related angles.

For example, for x = 30°.
The family of related angles would be:
30°, 150°, 210°, 330°

These angles are related since:
$\sin 30^\circ = 0.5$
$\sin 150^\circ = 0.5$
$\sin 210^\circ = -0.5$
$\sin 330^\circ = -0.5$

Note: The sine of these angles have the same numerical value.

Equations

Example A: $\sin x^\circ = 0.423 \quad (0 \le x \le 360)$ $x = \sin^{-1}(0.423)$ $x = 25^\circ$ (R.A) $x = (0 + 25)^\circ, (180 - 25)^\circ$ $x = 25^\circ, 155^\circ$	**Step 1: Consider 0.423** **Step 2: We know that we can find the other 3 angles in the family 155°, 205°, 335°** **Step 3: We only want the angles which will give +ve answers for sin.**
Example B: $\cos x^\circ = -0.584 \quad (0 \le x \le 360)$ $x = \cos^{-1}(0.584)$ $x = 54.3^\circ$ (R.A) $x = (180 - 54.3)^\circ, (180 + 54.3)^\circ$ $x = 125.7^\circ, 234.3^\circ$	**Step 1: Consider 0.584 (ignore -ve)** **Step 2: We know that we can find the other 3 angles in the family 125.7°, 234.3°, 305.7°** **Step 3: We only want the angles which will give -ve answers for cos.**

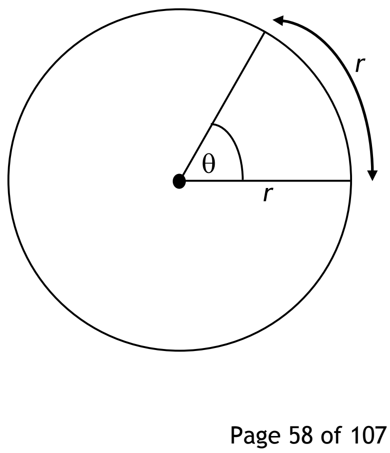
Radians

If we draw a circle and make a sector with an arc of exactly one radius long, then the angle at the centre of the sector is called a **radian.**

Remember that Circumference = $\pi D = 2\pi r$. This means that there are 2π radians in a full circle.

$360^\circ = 2\pi$ radians

$180^\circ = \pi$ radians



Example 1: Convert:

a) 90° to radians

b) 60° to radians

c) 225° to radians

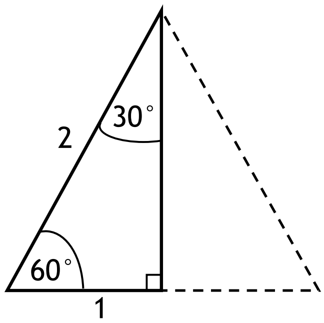
d) $\frac{\pi}{4}$ radians to degrees

e) $\frac{4\pi}{3}$ radians to degrees

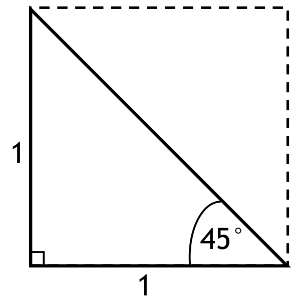
f) $\frac{11\pi}{6}$ radians to degrees

Exact Values

Consider the following triangles:



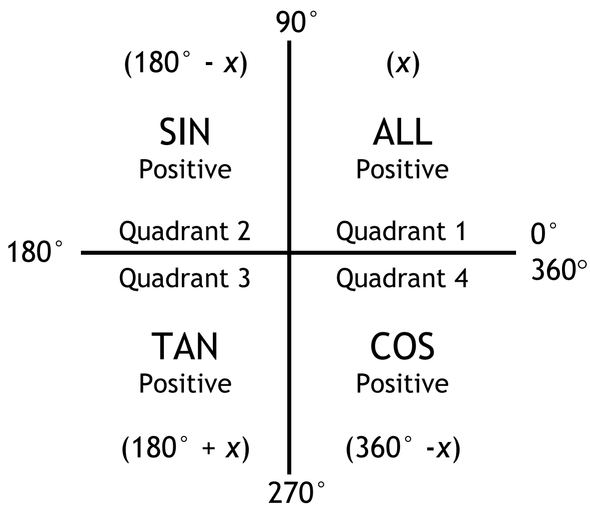
A right-angled triangle made by halving an equilateral triangle of side 2 units



A right-angled triangle made by halving an square of side 1 unit

Once we have found the lengths of the missing sides (by Pythagoras' Theorem), the following table of values can be constructed:

	0°	30°	45°	60°	90°
	0	$\left(\frac{\pi}{6}\right)$	$\left(\frac{\pi}{4}\right)$	$\left(\frac{\pi}{3}\right)$	$\left(\frac{\pi}{2}\right)$
Sin					
Cos					
Tan					



Example 2: State the exact values of:

a) $\sin 150^\circ$

b) $\tan 315^\circ$

c) $\cos \frac{7\pi}{6}$

Addition Formulae

Finding the value of a compound angle is not quite as simple as adding together the values of the component angles, e.g. $\sin 90° \neq \sin 60° + \sin 30°$. The following formulae must be used:

$\sin(A + B) = \sin A\cos B + \cos A\sin B$	$\sin(A - B) = \sin A\cos B - \cos A\sin B$

Example 3: Expand each of the following:

a) $\sin(X + Y)$

b) $\sin(Q + 3P)$

Example 4: Find the exact value of $\sin 75°$.

Example 5: A and B are acute angles where $\tan A = \frac{12}{5}$ and $\tan B = \frac{3}{4}$. Find the value of $\sin(A + B)$.

Example 6: Expand each of the following:

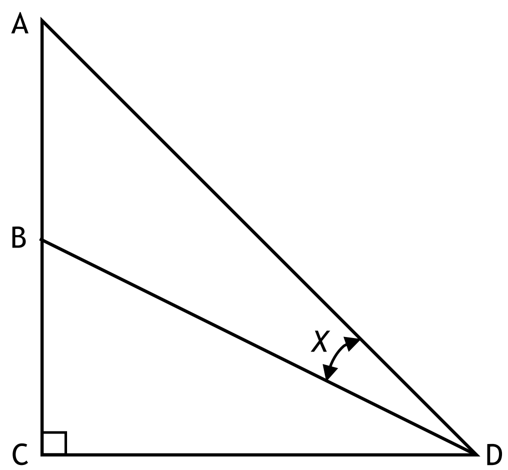
a) $\sin(\alpha - \beta)$

b) $\sin\left(2B - \frac{2\pi}{3}\right)$

Example 7: In the diagram opposite:

AC = CD = 2 units, and AB = BC = 1 unit.

Show that $\sin X$ is exactly $\frac{1}{\sqrt{10}}$.



cos(A + B) and cos (A – B)

cos(A + B) = cosAcosB - sinAsinB

cos(A - B) = cosAcosB + sinAsinB

Example 8: Expand the following:

a) cos(X – Y)

b) cos(X + 315)°

Example 9:

a) Show that $\frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12}$

b) Hence find the exact value of $\cos\frac{\pi}{12}$

To summarise:

$$\sin\left(A\pm B\right)=\sin A\cos B\pm\cos A\sin B$$

$$\cos\left(A\pm B\right)=\cos A\cos B\mp\sin A\sin B$$

Trigonometric Identities

NOTE: these are important formulae which are **not provided** in the exam paper formula sheets!

$$\frac{\sin x^\circ}{\cos x^\circ} = \tan x^\circ$$

$$\sin^2 x^\circ + \cos^2 x^\circ = 1$$

Note that due to the second formula, we can also say that:

$$\cos^2 x^\circ = 1 - \sin^2 x^\circ$$ AND $$\sin^2 x^\circ = 1 - \cos^2 x^\circ$$

To prove that an identity is true, we need to show that the expression on the left hand side of the equals sign can be changed into the expression on the right hand side.

Example 10: Prove that:

a) $\cos^4 \alpha - \sin^4 \alpha = \cos^2 \alpha - \sin^2 \alpha$

b) $\tan 3\theta + \tan\theta = \dfrac{\sin 4\theta}{\cos\theta\cos 3\theta}$

c) $\tan x - \dfrac{1}{\tan x} = \dfrac{2\sin^2 x - 1}{\sin x \cos x}$

Double Angle Formulae

sin2A = sin(A + A)

=

cos2A = cos(A + A)

=

Since $\cos^2 x° = 1 - \sin^2 x°$ and $\sin^2 x° = 1 - \cos^2 x°$, we can further expand the formula for cos2A:

$\cos 2A = \cos^2 A - \sin^2 A$

=

$\cos 2A = \cos^2 A - \sin^2 A$

=

To summarise:

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

Example 11: Express the following using double angle formulae:

a) sin2X

b) sin6Y

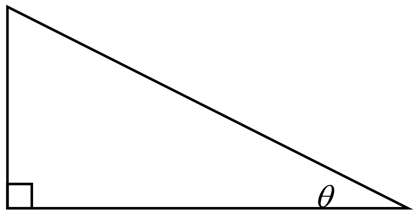
c) cos2X (sine version)

d) cos8H (cosine version)

e) sin5Q

f) $\cos\theta$ (cos and sin version)

Example 12: $\sin\theta = \dfrac{2}{\sqrt{13}}$, where θ is an acute angle. Find the exact values of:



a) sin(2θ)

b) cos(2θ)

Example 13: Prove that $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$

Solving Complex Trig Equations

Trig equations can also often involve (i) powers of sin, cos or tan, and (ii) multiple and/or compound angles.

Example 14: Solve $4\cos^2 x - 3 = 0$ for $0 \le x \le 2\pi$

Trig equations can also be written in forms which resemble quadratic equations: to solve these, treat them as such, and solve by factorisation.

Example 15: Solve $6\sin^2 x° - \sin x° - 2 = 0$ for $0 \le x \le 360°$

If the equation contains a multiple angle term, solve as normal (paying close attention to the range of values of x).

Example 16: Solve $\sqrt{3}\tan(2x - 135)° = 1$ for $0 \le x \le 360°$

To solve trig equations with combinations of double- and single-angle angle terms:

- **Rewrite the double angle term using the formulae on Page 59**
- **Factorise**
- **Solve each factor for** x

When the term is $\cos 2X$, the version of the double angle formula we use depends on the other terms in the equation: use $2\cos^2 x - 1$ if the other term is $\cos x$; $1 - 2\sin^2 x$ if the other term is $\sin x$.

Example 17: Solve $\sin 2x° - 2\sin x° = 0$, $0 \le x \le 360°$

Example 18: Solve $2\cos 2x - 7\cos x = 0$, $0 \leq x \leq 2\pi$

Formulae for $\cos^2 x$ and $\sin^2 x$

Rearranging the formulae for $\cos 2x$ allows us to obtain the following formulae for $\cos^2 x$ and $\sin^2 x$

$$\cos^2 x = \frac{1}{2}\left(1 + \cos 2x\right)$$

$$\sin^2 x = \frac{1}{2}\left(1 - \cos 2x\right)$$

Example 19: Express each of the following without a squared term:

a) $\cos^2\theta$

b) $\sin^2 3X$

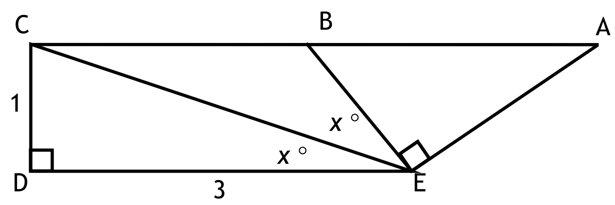
c) $\sin^2(\frac{[illegible]}{2})$

Past Paper Example 1: In the diagram,

$\angle DEC = \angle CEB = x°$, and $\angle CDE = \angle BEA = 90°$.

CD = 1 unit; DE = 3 units.

By writing $\angle DEA$ in terms of x, find the exact value of $\cos(D\hat{E}A)$.



Past Paper Example 2: Find the points of intersection of the graphs of $y = 3\cos 2x^\circ + 2$ and $y = 1 - \cos x^\circ$ in the interval $0 \le x \le 360^\circ$.

Past Paper Example 3: Solve algebraically the equation

$$\sin 2x = 2\cos^2 x \qquad \text{for } 0 \le x \le 2\pi$$