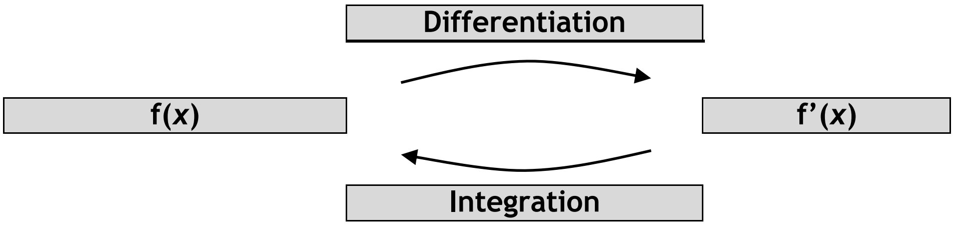
Calculus 2: Integration

The reverse process to differentiation is known as **integration.**



As it is the opposite of finding the derivative, the function obtained by integration is sometimes called the **anti-derivative,** but is more commonly known as the **integral**, and is given the sign $\int$.

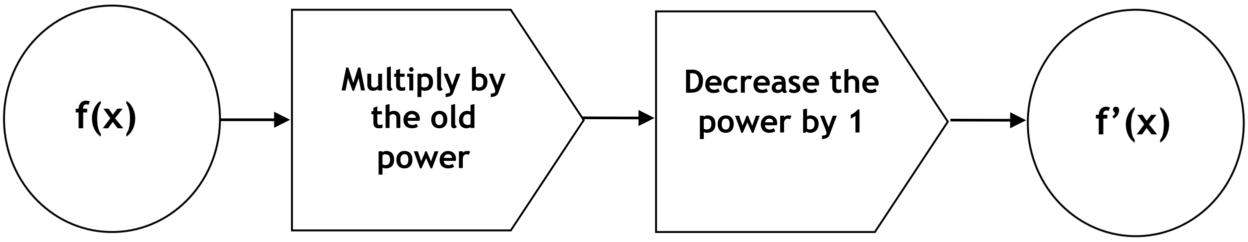
If f(x) = x^n, then $\int x^n dx$ is "the integral of x^n with respect to x "

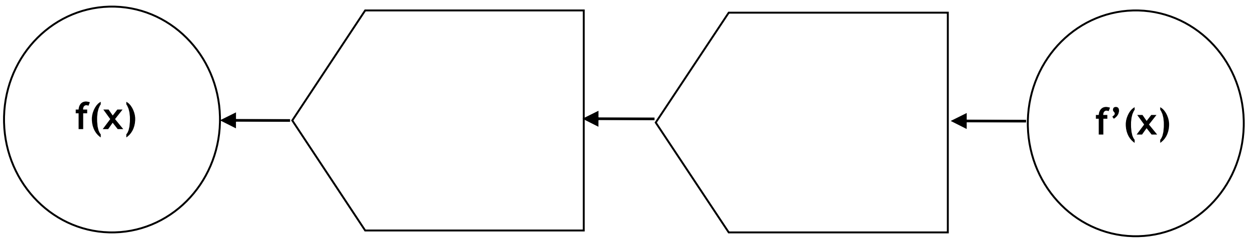
Indefinite Integrals and the Constant of Integration

Consider the three functions a(x) = $3x^2 + 2x + 5$, b(x) = $3x^2 + 2x - 8$ and c(x) = $3x^2 + 2x - \frac{13}{4}$.

In each case, the derivative of the function is the same, i.e. $6x + 2$. This means that $\int (6x+2)dx$ has more than one answer. Because there is more than one answer, we say that this is an **indefinite integral**, and we must include in the answer a constant value C, to represent the 5, -8, $-\frac{13}{4}$ etc which we would need to distinguish a(x) from b(x) from c (x) etc.

To find the integral of a function, we do the **opposite** of what we would do to find the derivative:





In general: $$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C \quad (n \neq -1)$$

IN **tegration** ***IN*** **creases the power!**

1. **Write as ax^n**
2. **Increase the power by 1**
3. **Divide by the new power**

Example 1: Find **(remember "+C"):**

a) $\int 2x\,dx$ **b)** $\int 4t^2\,dt$ **c)** $\int (3x^5 - 4)dx$

d) $\int \frac{3}{g^4} dg \quad (g \neq 0)$

e) $\int 6\sqrt[5]{p^3}\, dp$

f) $\int \frac{4y-3}{y^{\frac{2}{3}}} dy \quad (y \neq 0)$

The Definite Integral

A **definite integral** of a function is the difference between the integrals of f(x) at two values of x . Suppose we integrate f(x) and get F(x). Then the integral of f(x) when x = a would be F(a), and the integral when x = b would be F(b).

The definite integral of f(x), with respect to x , between a and b, is written as:

$$\int_a^b \mathbf{f(x)dx = F(b) - F(a)} \qquad \textbf{(where b > a)}$$

For example, the integral of f(x) = 2x^2 – 4 between the values x = -3 and x = 5 is written as $\int_{-3}^{5} (2x^2 - 4)\,dx$ and reads "the integral from -3 to 5 of 2x^2 – 4 with respect to x ".

Note: definite integrals do NOT include the constant of integration!

$$\int_a^b f(x) = [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

Example 2: Evaluate $\int_{-1}^{3} (2x-1)\,dx$

To find a definite integral:

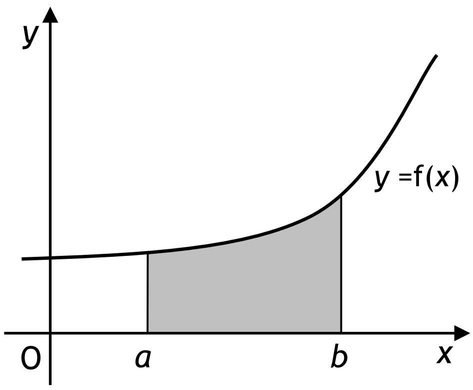
- **prepare the function for integration**
- **integrate as normal, but write inside square brackets with the limits to the right**
- **sub each limit into the integral, and subtract the integral with the lower limit from the one with the higher limit**

Example 3: Evaluate $\int_0^2 (p+1)(p-1)\,dp$

Example 4: Evaluate $\int_1^{\sqrt{3}} (x^2 - 2x)\,dx$

Example 5: Find the value of g such that $\int_{-2}^{g}(6x+5)\,dx = 6$.

Area Between a Curve and the *x* - axis.



In the diagram opposite, the area of the shaded section can be obtained by finding the area under the graph from O to b, and subtracting the area from O to a.

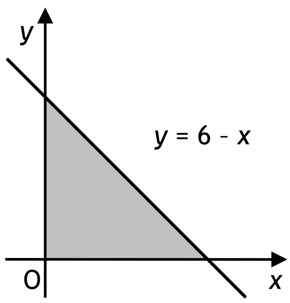
The value of each of these areas can be determined by integrating the function and substituting b or a respectively.

The area enclosed by the curve y = f(x), the lines y = a, y = b and the x - axis is equal to the definite integral of f(x) between a and b

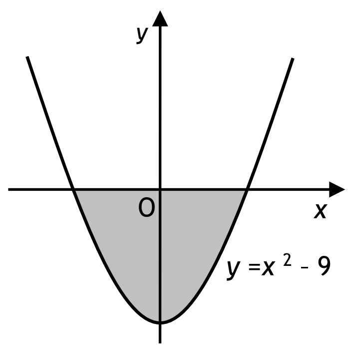
i.e. Area = $\int_a^b f(x)\,dx$

Example 6: For each graph below,
(i) write down the integrals which describe the shaded regions
(ii) calculate the area of the shaded region

a)



b)

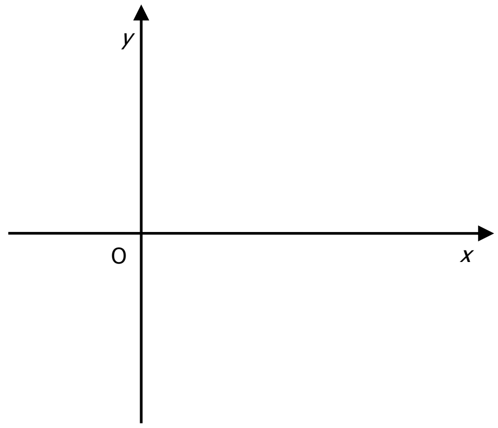


NOTE: Example 6b shows that areas UNDER the *x* - axis give NEGATIVE values!

Example 7:

a) Evaluate $\int_{-1}^{7}(2x-6)\,dx$

b) (i) Sketch below the area described by the integral $\int_{-1}^{7}(2x-6)\,dx$.

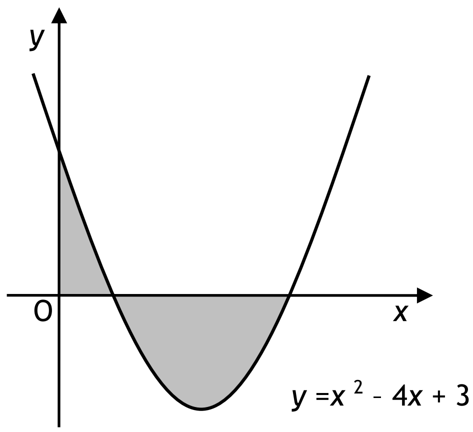


The answers for 5a and 5b do not match! This is because the area below the axis and the area above cancel each other out (as in 4b, areas below the *x* – axis give negative values).

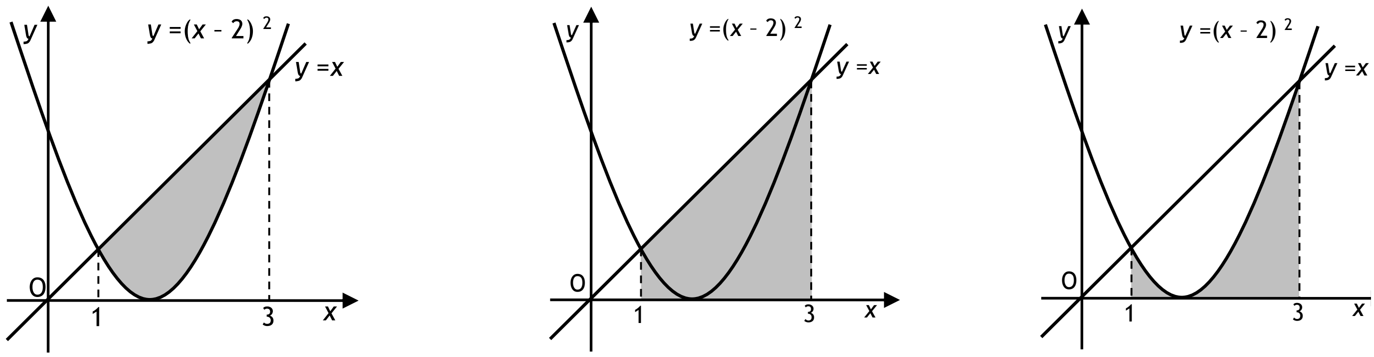
To find the area between a curve and the x-axis:

1. **Determine the limits which describe the sections above and below the axis**
2. **Calculate areas separately**
3. **Find the total, *IGNORING THE NEGATIVE VALUE OF THE SECTION BELOW THE AXIS*.**

Example 8: Determine the area of the regions bounded by the curve $y = x^2 - 4x + 3$ and the *x* – and *y* – axes.

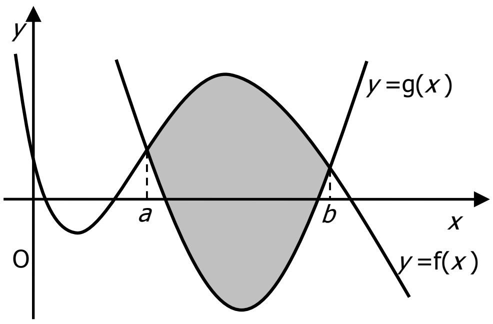


Consider the area bounded by the curves $y = (x - 2)^2$ and $y = x$.



Area = $\int_1^3 x\,dx$ - $\int_1^3 (x-2)^2\,dx$

The diagrams above show that the area between the curves is equal to the area between the top function (x) and the x – axis MINUS the area between the bottom curve ($(x - 2)^2$) and the x – axis.



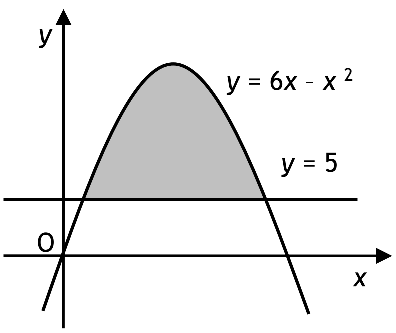
The area between the curves $y = f(x)$ and $y = g(x)$ (which meet at the points where $x = a$ and $x = b$) is given by:

$$\mathbf{A = \int_a^b (f(x) - g(x))dx}$$

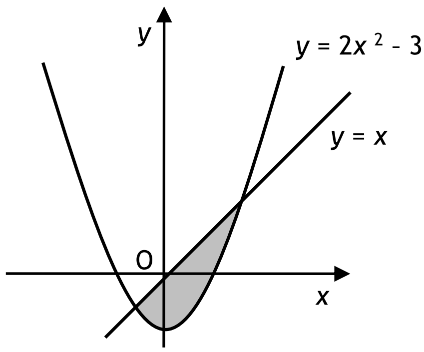
where:
- $f(x)$ is the TOP function and $g(x)$ is the BOTTOM
- $b > a$

Example 9: Write down the integrals used to determine the areas shown below:

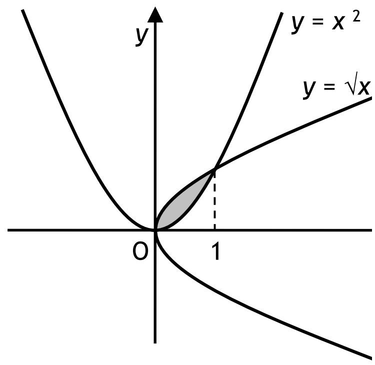
a)



b)



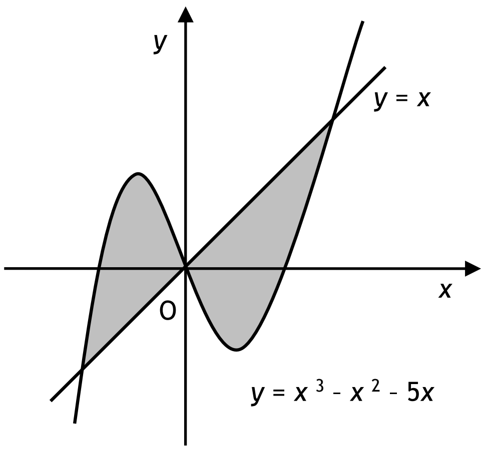
c)



To find the area between two curves:

1. **Make a sketch (if one has not been given)**
2. **Find points of intersection (make *y* = *y* and solve)**
3. **Subtract the bottom function from the top function, *PUTTING THE BOTTOM FUNCTION IN BRACKETS!***
4. **Integrate**

Example 10: Find the area enclosed between the curve $y = x^3 - x^2 - 5x$ and the line $y = x$



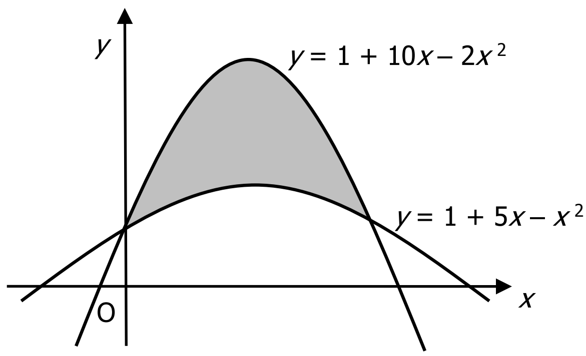
Differential Equations

If we know the derivative of a function (e.g. $f'(x) = 6x^2 - 3$), we can obtain a formula for the original function by integration. This is called a **differential equation**, and gives us the function in terms of *x* and C (which we can then evaluate if we have a point on the graph of the function).

Example 11: The gradient of a tangent to the curve of $y = f(x)$ is $24x^2 + 10x$, Express *y* in terms of *x*, given than the graph of $y = f(x)$ passes through the point (-1, -10).

Past Paper Example 1: Evaluate $\int_1^9 \frac{x+1}{\sqrt{x}}\,dx$

Past Paper Example 2: Find area enclosed between the curves $y = 1 + 10x - 2x^2$ and $y = 1 + 5x - x^2$.

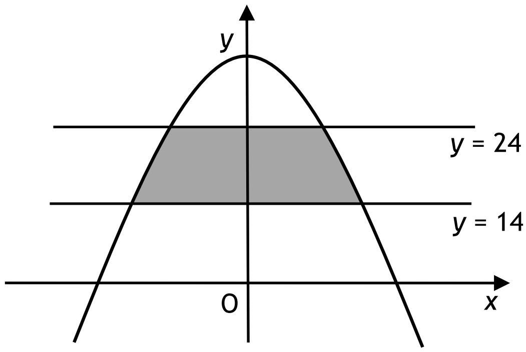


Past Paper Example 3: The parabola shown in the diagram has equation

$$y = 32 - 2x^2.$$

The shaded area lies between the lines $y = 14$ and $y = 24$

Calculate the shaded area.



Applications Unit Topic Checklist (Unit Assessment Topics in Bold)			
Topic		Questions	Done?
Straight Line	**Gradients (inc. m = tanθ)**	Exercise 1A, Q 8 – 10; Exercise 1B, p 4, Q 4, 5	Y/N
	Perpendicular Gradients	Exercise 1D, Q 1 – 4, 7	Y/N
	Equations of straight lines	Exercise 1E, Q 1, 3, 7, 8 ($y = mx + c$)	Y/N
		Exercise 1F, Q 1, 2 ($Ax + By + C = 0$)	Y/N
		Exercise 1G, Q 2, 3 ($y - b = m(x - a)$)	Y/N
	Collinearity	Exercise 1B, Q 1 – 3, 9	Y/N
	Perpendicular bisectors	Exercise 1I, Q 1, 2; Exercise 1N, Q 5	Y/N
	Altitudes	Exercise 1K, Q 1, 5; Exercise 1N, Q 1 – 3	Y/N
	Medians	Exercise 1M, Q 1, 3; Exercise 1N, Q 4	Y/N
	Distance Formula	Exercise 12B, Q 1	Y/N
Recurrence Relations	**Finding terms**	Exercise 5D, Q 1 – 3	Y/N
	Creating & using formulae	Exercise 5C, Q 5 – 11	Y/N
	Finding a limit	Exercise 5H, Q 1 – 3	Y/N
		Exercise 5H, Q 4 – 10; Exercise 5L, p 83, Q 2, 4	Y/N
	Solving to find *a* and *b*	Exercise 5I, Q 1, 2	Y/N
		Exercise 5I, Q 3, 4	Y/N
The Circle	**Circles centred on O**	Exercise 12D, Q 1 – 3	Y/N
	$(x - a)^2 + (y - b)^2 = r^2$	Exercise 12F, Q 1 – 3, 10	Y/N
	General equation	Exercise 12H, Q 1, 4, 12 – 15; Exercise 12M, Q 1, 7	Y/N
	Intersection of lines & circles	Exercise 12J, Q 3	Y/N
	Tangency	Exercise 12K, Q 2, 6; Exercise 12M, Q 4, 8	Y/N
	Equations of tangents	Exercise 12L, Q 1 – 4	Y/N
Calculus	**Optimisation**	Exercise 6Q, Q 1, 2, 4	Y/N
		Exercise 6R, Q 1, 5; Exercise 6S, Q 19	Y/N
	Area under a curve	Exercise 9K, Q 1; Exercise 9N, Q 1, 3, 4	Y/N
	Area between two curves	Exercise 9P, Q 1, 2, 4; Exercise 9R, Q 7, 11	Y/N
	Differential Equations	Exercise 9Q, Q 2, 3; Exercise 9R, Q 14, 15	Y/N