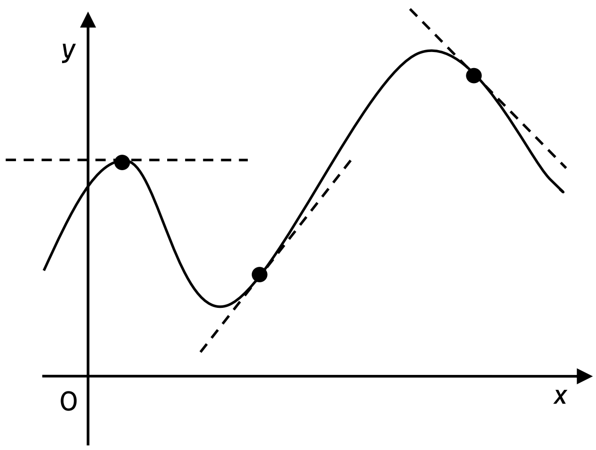
Calculus 1: Differentiation

In the chapter on straight lines, we saw that the gradient of a line is a measure of how quickly it increases (or decreases) at a constant rate.

This is easy to see for linear functions, but what about quadratic, cubic and higher functions? As these functions produce curved graphs, they do **not** increase or decrease at a constant rate.

For a function f(x), the rate of change at any point on the function can be found by measuring the gradient of a **tangent** to the curve at that point.



The rate of change at any point of a function is called the **derived function** or the **derivative.**

Finding the rate of change of a function at a given point is part of a branch of maths known as **calculus.**

For function f(x) or the graph y = f(x), the derivative is written as:

f'(x) ("f dash x")

OR

$\frac{dy}{dx}$ **("dy by dx")**

Derivative = Rate of Change of the Function = Gradient of the Tangent to the Curve

The Derivative of f(x) = ax^n

Example 1: Find the derivative of f(x) = x^2

To find the derivative of a function:

1. **Make sure it's written in the form y = ax^n**
2. **Multiply by the power**
3. **Decrease the power by one**

Example 2: f(x) = 2x^3. Find f'(x).

This means:

At x = 1, the gradient of the tangent to 2x^3 =

At x = -2, the gradient of the tangent to 2x^3 =

If f(x) = ax^n, then f'(x) = nax^{n-1}

The *__DE__* rivative *__DE__* creases the power!

To find the derivative of f(x):

- f(x) MUST be written in the form f(x) = ax^n
- Rewrite to eliminate fractions by using negative indices
- Rewrite to eliminate roots by using fractional indices

Revision from National 5

Example 3: Write with negative indices:

a) $\frac{2}{x^2}$ b) $\frac{1}{4x^5}$ c) $\frac{3}{5x}$

Example 4: Write in index form:

a) $\sqrt{x}$ b) $\sqrt[3]{x^2}$ c) $\frac{2}{3\sqrt{x^7}}$

Example 5: For each function, find the derivative.

a) $f(x) = x^{35}$

b) $g(x) = -x^{-3}$ $(x \neq 0)$

c) $p(x) = \dfrac{1}{\sqrt{x}}$ $(x > 0)$

d) $y = 12x^5 + 3x^2 - 2x + 9$

e) $y = \dfrac{1}{3\sqrt{x}}$ $(x > 0)$

f) $y = \left(\sqrt{x} - 2\right)^2$ $(x \geq 0)$

Example 6: Find the rate of change of each function:

a) $f(x) = \dfrac{x^5 - 6x^3}{x^2}$

b) $y = \dfrac{(x+3)^2}{x^{2/3}}$

c) $f(x) = \dfrac{x^5 - 3x}{2x^3}$

Points to note:

- **Number terms disappear (e.g. if $f(x) = 5$, $f'(x) = 0$)**
- ***x* - terms leave their coefficient (e.g. if $f(x) = 135x$, $f'(x) = 135$)**
- **Give your answer back in the same form as the question**

Equation of a Tangent to a Curve

Example 7: Find the equation of the tangent to the curve $y = x^2 - 2x - 15$ when $x = 4$.

To find the equation of a tangent to a curve:

- **Find the point of contact (sub the value of *x* into the equation to find *y*)**
- **Find $\frac{dy}{dx}$**
- **Find *m* by substituting *x* into $\frac{dy}{dx}$**
- **Use $y - b = m(x - a)$**

Example 8:

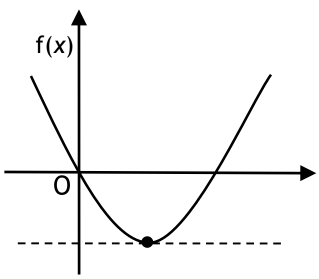
a) Find the gradient of the tangent to the curve $y = x^3 - 2x^2$ at the point where $x = \frac{7}{3}$.

b) Find the other point on the curve where the tangent has the same gradient.

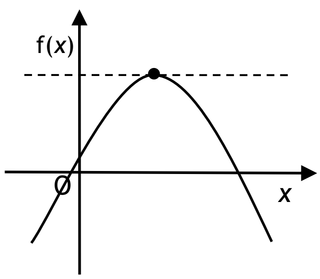
Example 9: Find the point of contact of the tangent to the curve with equation $y = x^2 + 7x + 3$ when the gradient of the tangent is 9.

Stationary Points and their Nature

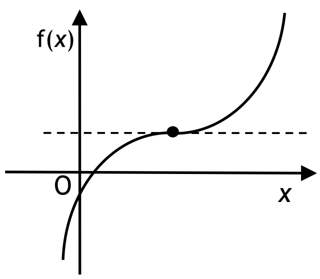
Any point on a curve where the tangent is horizontal (i.e. the gradient or $\frac{dy}{dx} = 0$) is commonly known as a **stationary point.** There are four types of stationary point:



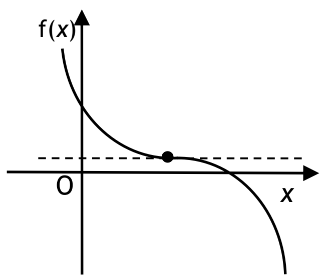
Minimum Turning Point



Maximum Turning Point



Rising Point of Inflection



Falling Point of Inflection

To locate the position of stationary points, we find the derivative, make it equal zero, and solve for x. To determine their type (or **nature**), we must use a **nature table**.

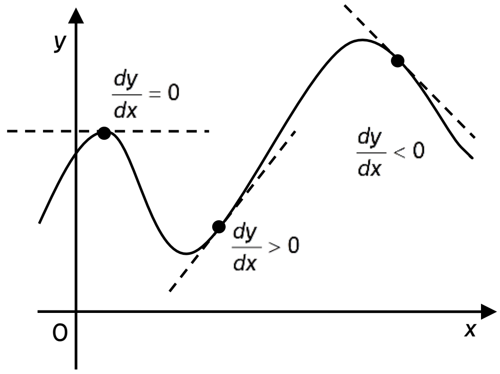
Example 10: Find the stationary points of the curve $y = 2x^3 - 12x^2 + 18x$ and determine their nature.

For any curve,

if $\frac{dy}{dx} > 0$, then y is **increasing**

if $\frac{dy}{dx} < 0$, then y is **decreasing**

if $\frac{dy}{dx} = 0$, then y is **stationary**



If a function is **always** increasing (or decreasing), it is said to be **strictly** increasing (or decreasing).

Example 11: State whether the function $f(x) = x^3 - x^2 - 5x + 2$ is **increasing**, **decreasing** or **stationary** when:

a) $x = 0$

b) $x = 1$

c) $x = 2$

Example 12: Show algebraically that the function $f(x) = x^3 - 6x^2 + 12x - 5$ is never decreasing.

Example 13: Find the intervals in which the function $f(x) = 2x^3 - 6x^2 + 5$ is increasing and decreasing.

Curve Sketching

To accurately sketch and annotate the curve obtained from a function, we must consider:

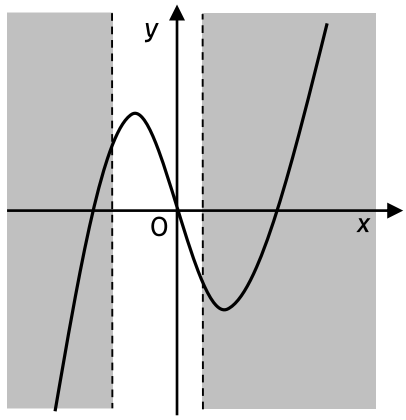
1. ***x* – and *y* – intercepts**
2. **Stationary points and their nature**

Example 14: Sketch and annotate fully y = $x^3(4 - x)$

Closed Intervals

Sometimes, we may only be interested in a small section of the curve of a function. To find the maximum and minimum values of a function in a given interval, we find stationary points as normal, but we also need to consider the value of the function at the ends of the interval.

Example 15: Find the greatest and least values of y = $x^3 - 12x$ on the interval $-3 \leq x \leq 1$.



Note: In a closed interval. The maximum and minimum values of a function occur either at a Stationary Point within the interval or at the end point of the interval.

Differentiation in Context: Optimisation

Differentiation can be used to find the maximum or minimum values of things which happen in real life. Finding the maximum or minimum value of a system is called **optimisation.**

Example 16: A carton is in the shape of a cuboid with a rectangular base and a volume of 3888cm^3.

The surface area of the carton can be represented by the formula $A(x) = 4x^2 + \dfrac{5832}{x}$.

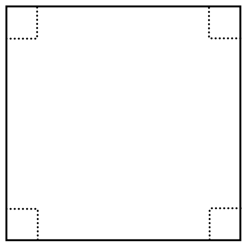
Find the value of x such that the surface area is a minimum.

In exams, optimisation questions almost always consist of two parts: part one asks you to **show** that a situation can be described using an algebraic formula or equation, whilst part two asks you to use the given formula to find a maximum or minimum value by differentiation.

Leave part 1 of an optimisation question until the end of the exam (if you have time), as they are almost always (i) more difficult than finding the stationary point and (ii) worth fewer marks.

Remember that part 2 is just a well-disguised "find the minimum/maximum turning point of this function" question!

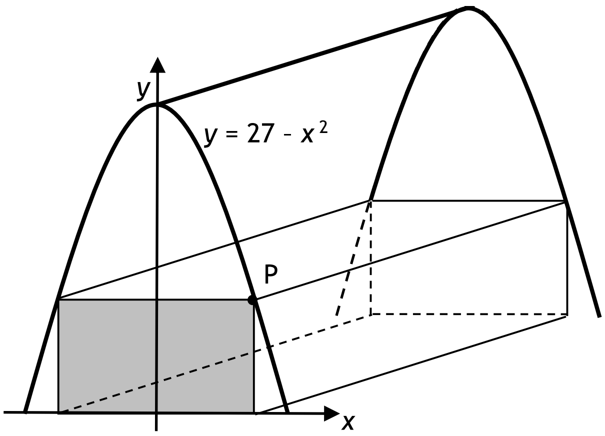
Example 17: A square piece of card of side 30cm has a square of side x cm cut from each corner. An open box is formed by turning up the sides.



a) Show that the volume, V , of the box may be expressed as $900x - 120x^2 + 4x^3$

b) Find the maximum volume of the box.

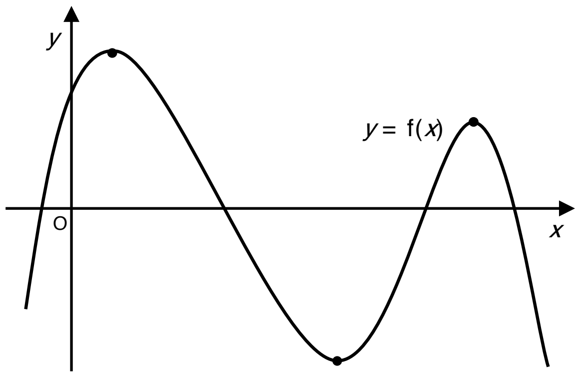
Example 18: An architect has designed a new open-plan office building using two identical parabolic support beams spaced 25m apart as shown below. The front beam, relative to suitable axes, has the equation $y = 27 - x^2$. The inhabited part of the building is to take the shape of a cuboid.



a) By considering the point P in the corner of the front face of the building, show that the area of this face is given by $\boldsymbol{A(x) = 54x - 2x^3}$.

b) Find the maximum volume of the inhabited section of the building.

Graph of the Derived Function



From the graph of y = f(x), we can obtain the graph of y = f'(x) by considering its stationary points. On the graph of y = f'(x), the y-coordinate comes from the derivative of y = f(x).

1. Draw a set of axes directly under a copy of y = f(x).

2. Locate the stationary points.

3. At SP's, f'(x) = 0, so the y coordinate of f'(x) = 0 on the new graph.

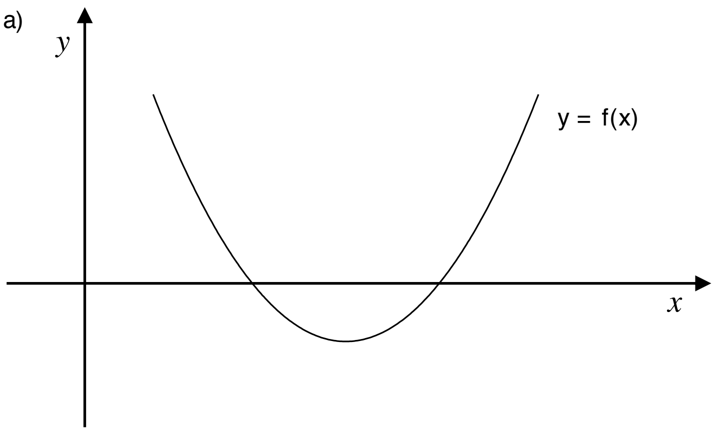
4. Where f(x) is increasing, f'(x) is **above** the x – axis.

5. Where f(x) is decreasing, f'(x) is **below** the x – axis.

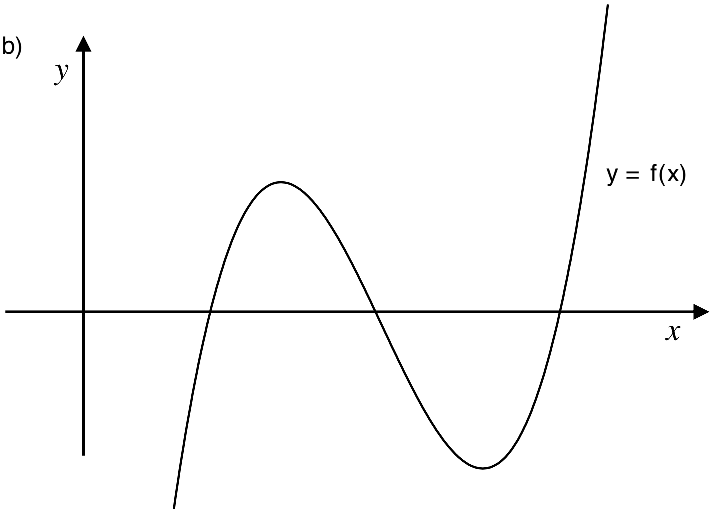
6. Draw a smooth curve which fits this information.

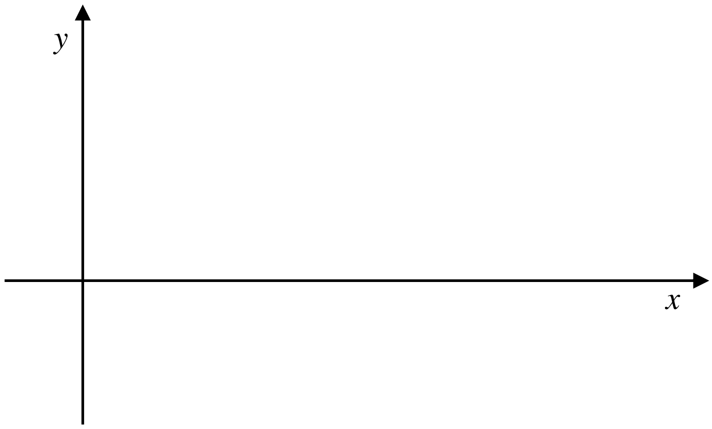
Example 19: For the graphs below. Sketch the corresponding derived graphs of y = f'(x)

a)



b)



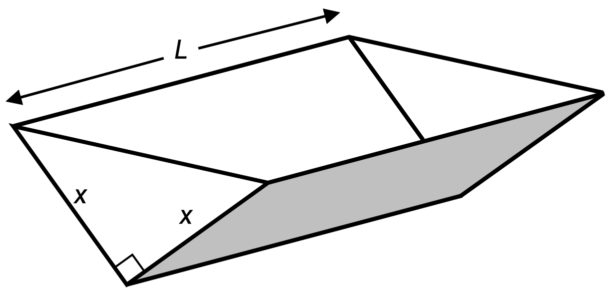


Past Paper Example 1: A curve has equation $y = x^4 - 4x^3 + 3$. Find the position and nature of its stationary points.

Past Paper Example 2: Find the equation of the two tangents to the curve $y = 2x^3 - 3x^2 - 12x + 20$ which are parallel to the line $48x - 2y = 5$.

Past Paper Example 3: An open water tank, in the shape of a triangular prism, has a capacity of 108 litres. The tank is to be lined on the inside in order to make it watertight.

The triangular cross-section of the tank is right-angled and isosceles, with equal sides of length x cm. The tank has a length of L cm.



a) Show that the surface area to be lined, A cm^2, is given by $A(x) = x^2 + \dfrac{432000}{x}$

b) Find the minimum surface area of the tank.

Past Paper Example 4: A function is defined on the domain $0 \leq x \leq 3$ by $f(x) = x^3 - 2x^2 - 4x + 6$.

Determine the maximum and minimum values of f.