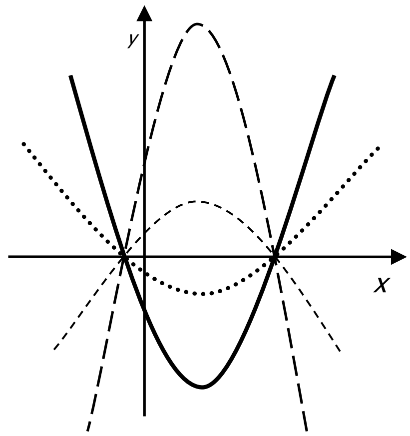
Quadratic Functions

Finding the Equation of a Quadratic Function From Its Graph: $y = k(x - a)(x - b)$

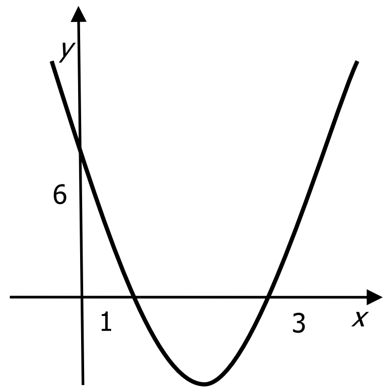
If the graph of a quadratic function has roots at $x = -1$ and $x = 5$, a reasonable guess at its equation would be $y = x^2 - 4x - 5$, i.e. from $y = (x + 1)(x - 5)$.

However, as the diagram shows, there are many parabolas which pass through these points, all of which belong to the **family** of functions $\boldsymbol{y = k(x + 1)(x - 5)}$.

To find the equation of the original function, we need the roots and one other point on the curve (to allow us to determine the value of k).



Example 1: State the equation of the graph below in the form $y = ax^2 + bx + c$.



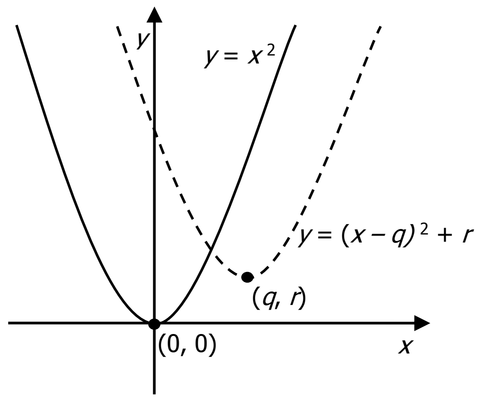
Completing the Square (Revision)

The diagram shows the graphs of two quadratic functions.

If the graph of $y = x^2$ is shifted q units to the right, followed by r units up, then the graph of $y = (x - q)^2 + r$ is obtained.

As the turning point of $y = x^2$ is (0, 0), it follows that the new curve has a turning point at **(q , r).**

A quadratic equation written as $y = p(x - q)^2 + r$ is said to be in the **completed square form.**



Example 2: (i) Write the following in the form $y = (x + q)^2 + r$ and find the minimum value of y.

(ii) Hence state the minimum value of y and the corresponding value of x .

a) $y = x^2 + 6x + 10$

b) $y = x^2 - 3x + 1$

Completing the Square when the x^2 Coefficient ≠ 1

Example 3: Write $y = 3x^2 + 12x + 5$ in the form $y = p(x + q)^2 + r$.

Example 4: Write $y = 5 + 12x - x^2$ in the form $y = p - (x + q)^2$.

Example 5:

a) Write $y = x^2 - 10x + 28$ in the form $y = (x + p)^2 + q$.

b) Hence find the maximum value of $\dfrac{18}{x^2 - 10x + 28}$

Solving Quadratic Equations via Completing the Square

Quadratic equations which do not easily factorise can be solved in two ways: (i) completing the square, or (ii) using the quadratic formula. In fact, both methods are essentially the same, as the quadratic formula is obtained by solving $y = ax^2 + bx + c$ via completing the square.

Example 6: State the **exact** values of the roots of the equation $2x^2 - 4x + 1 = 0$ by:

a) using the quadratic formula

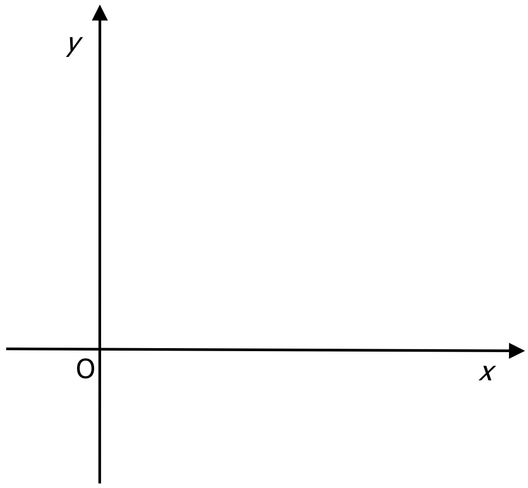
b) completing the square

Solving Quadratic Inequations

Quadratic inequations are easily solved by making a sketch of the equivalent quadratic function, and determining the regions above or below the x – axis.

Example 7: Find the values of x for which: a) $2x^2 - 7x + 6 > 0$ b) $2x^2 - 7x + 6 < 0$

First, sketch $y = 2x^2 - 7x + 6$



Roots of Quadratic Equations and The Discriminant (Revision)

For $y = ax^2 + bx + c$, $b^2 - 4ac$ is known as the **discriminant.**

- **$b^2 - 4ac > 0$ gives real, unequal roots**
- **$b^2 - 4ac = 0$ gives real, equal roots**
- **$b^2 - 4ac < 0$ gives NO real roots**

If $b^2 - 4ac$ gives a perfect square, the roots are RATIONAL
If $b^2 - 4ac$ does NOT give a perfect square, the roots are IRRATIONAL (i.e. surds)

Example 8: Determine the nature of the roots of the equation $4x(x - 3) = 9$

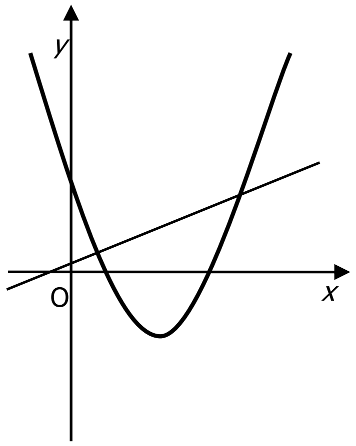
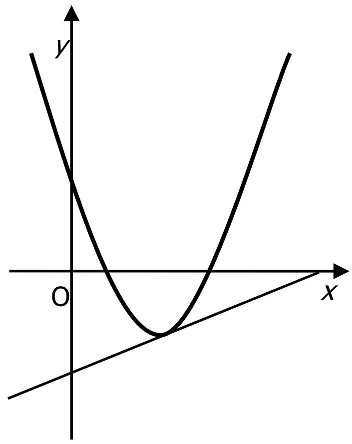
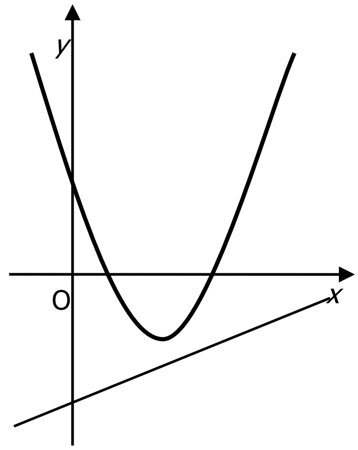
Example 9: Find the value(s) of p given that $2x^2 + 4x + p = 0$ has real roots.

Example 10: Find the value(s) of r given that $x^2 + (r - 3)x + r = 0$ has no real roots.

Tangents to Curves: Using the Discriminant

To find the points of contact between a line and a curve, we make the curve and line equations equal (i.e. make $y = y$) to obtain a quadratic equation, and solve to find the x-coordinates.

By finding the discriminant of this quadratic equation, we can work out **how many** points of contact there are between the line and the curve. There are 3 options:

 	 	
Two points of contact **2 distinct roots** $b^2 - 4ac > 0$	**One point of contact** **Equal roots** $b^2 - 4ac = 0$	**No points of contact** **No real roots** $b^2 - 4ac < 0$

The most common use for this technique is to show that a line is a tangent to a curve

Example 11: Show that the line $y = 3x - 13$ is a tangent to the curve $y = x^2 - 7x + 12$, and find the coordinates of the point of contact.

Example 12: Find **two** values of m such that $y = mx - 7$ is a tangent to $y = x^2 + 2x - 3$

Past Paper Example 1: Express $2x^2 + 12x + 1$ in the form $a(x + b)^2 + c$.

Past Paper Example 2: Given that $2x^2 + px + p + 6 = 0$ has no real roots, find the range of values for p.

Past Paper Example 3: Show that the roots of $(k - 2)x^2 - 3kx + 2k = -2x$ are **always** real.