Recurrence Relations

A **recurrence relation** is a rule which produces a sequence of numbers where each term is obtained from the previous one. Recurrence relations can be used to solve problems involving systems which grow or shrink by the same amount at regular intervals (e.g. the amount of money in a savings account which grows by 3.5% p/a, the volume of water left in a pool if 10% evaporates each day, etc).

Recurrence relations are generally written in one of two forms:

$$\mathbf{U_{n+1} = aU_n + b}$$

OR

$$\mathbf{U_n = aU_{n-1} + b}$$

In both cases, a term is found by multiplying the previous term by a constant *a*, then adding (or subtracting) another constant *b*.

U_n means the **n**th **term** in the sequence (i.e. U_7 would be the 7th term, etc). U_0 ("U zero") is the starting point of the sequence, e.g. the amount of money put into an account before interest is added.

Example 1: A sequence is defined by the recurrence relation $U_{n+1} = 3U_n + 2$, $U_0 = 4$.

Find the value of U_4.

Example 2: A sequence is defined by the recurrence relation $U_n = 4U_{n-1} - 3$, where $U_0 = a$.

Find an expression for U_2 in terms of a.

Finding a Formula

Recurrence relations can be used to describe situations seen in real life where a quantity changes by the same percentage at regular intervals. The first thing to do in most cases is find a formula to describe the situation.

Example: Jennifer puts £5000 into a high-interest savings account which pays 7.5% p/a. Find a recurrence relation for the amount of money in the savings account.

Solution: Starting amount = £5000
After 1 year: amount = starting amount + 7.5% (i.e. 107.5% of starting amount)
= 1.075 x starting amount

Recurrence relation is: $\mathbf{U_{n+1} = 1.075U_n}$ $\mathbf{(U_0 = 5000)}$

Example 3: Find a recurrence relation to describe:

a) The amount left to pay on a loan of £10000, with interest charged at 1.5% per month and fixed monthly payments of £250.

b) The amount of water in a swimming pool of volume 750,000 litres if 0.05% per day is lost to evaporation, but 350 litres extra is added daily.

Example 4: Bill puts lottery winnings of £120000 in a bank account which pays 5% interest p/a. After a year, he decides to spend £20000 per year from the money in the account.

a) Find a recurrence relation to describe the amount of money left each year.

b) How much money will there be in the account after five years?

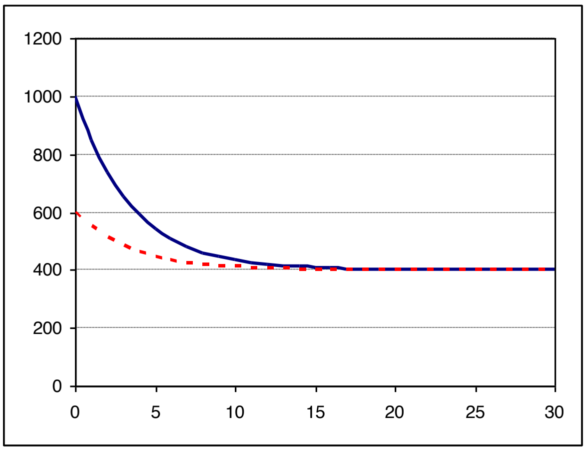
c) After how many years will Bill's money run out?

Limits of Recurrence Relations

Some recurrence relations produces sequences which tend towards a **limit**, i.e. as the number of terms increases, the sequence gets closer and closer to a certain value without actually meeting it.

The graph opposite shows two sequences generated by the recurrence relation $U_{n+1} = 0.75U_n + 100$, one where $U_0 = 1200$, the other where $U_0 = 600$.

In both cases, irrespective of the value of U_0, as U_n approaches infinity, the sequences tend to a limit of 400. How can we find the value of these limits without plotting a graph?



For $U_{n+1} = aU_n + b$, a limit exists when $-1 < a < 1$

If a limit exists, its value is independent of the value of U_0

Example 5: For the recurrence relation $U_{n+1} = 0.6U_n - 20$,

a) State whether a limit exists, and if so
b) Find the limit.

Example 6: A man plants some trees as a boundary between his house and the house next door. Each year, the trees are expected to grow by 0.5m. To counter this, he decides to trim them by 20% per year.

a) To what height will the trees eventually grow?

b) His neighbour is unhappy that the trees are too tall, and insists they grow no taller than 2m high. What is the **minimum** percentage they must be trimmed each year to meet this condition?

Solving Recurrence Relations to Find a and b

If we have three consecutive terms in a sequence, we can find the values of a and b in the recurrence relation which generated the sequence using simultaneous equations.

Example 7: A sequence is generated by a recurrence relation of the form $U_{n+1} = aU_n + b$.
In this sequence, $U_1 = 28$, $U_2 = 32$ and $U_3 = 38$. Find the values of a and b.

Past Paper Example: Marine biologists calculate that when the concentration of a particular chemical in a loch reaches 5 milligrams per litre (mg/L) the level of pollution endangers the lives of the fish.

A factory wishes to release waste containing this chemical into the loch, and supplies the Scottish Environmental Protection Agency with the following information:

1. The loch contains none of the chemical at present.
2. The company will discharge waste once per week which will result in an increase in concentration of 2.5 mg/L of the chemical in the loch.
3. The natural tidal action in the loch will remove 40% of the loch every week.

a) After how many weeks at this level of discharge will the lives of the fish become endangered?

b) The company offers to install a cleaning process which would result in an increase in concentration of only 1.75 mg/L of the chemical in the loch, and claim this will not endanger the lives of the fish in the long term.

Should permission be given to allow the company to discharge waste into the loch using this revised process? Justify your answer.