North Lanarkshire

Higher Mathematics Notes

2016/17

FORMULAE LIST

Circle:

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

The equation $(x-a)^2 + (y-b)^2 = r^2$ represents a circle centre (a, b) and radius r.

Scalar Product: $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$, where θ is the angle between **a** and **b**

or $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Trigonometric formulae: $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\begin{aligned} \sin 2A &= 2\sin A \cos A \\ \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned}$$

Table of standard derivatives:

$f(x)$	$f'(x)$
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$

Table of standard integrals:

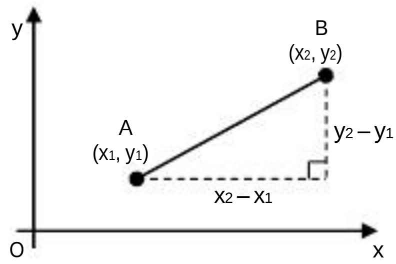
$f(x)$	$\int f(x)dx$
$\sin ax$	$-\frac{1}{a}\cos ax + C$
$\cos ax$	$\frac{1}{a}\sin ax + C$

The Straight Line

Revision from Nattional 5

The graph of $y = mx + c$ is a straight line, where m is the gradient and c is the y-intercept.

Gradient is a measure of the steepness of a line. The gradient of the line joining points A (x_1, y_1) and B (x_2, y_2) is given by:



$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 1: Find:

a) the gradient and y-intercept of the line $y = 2x + 5$

b) the equation of the line with gradient -4 and y-intercept $(0, -2)$

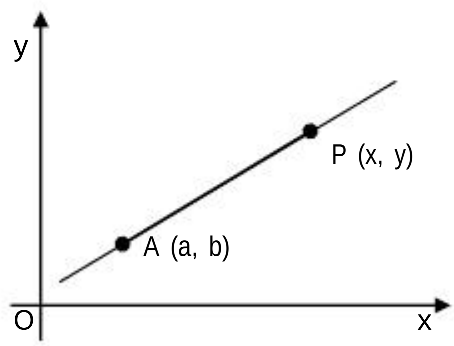
c) the gradient of the line joining P $(-2, 4)$ and Q $(3, -1)$

d) the gradient of the line $3y + 4x - 11 = 0$

Equation of a Straight Line: $y - b = m(x - a)$

Points A (a, b) and P (x, y) both lie on a straight line.

The gradient of the line $m = \frac{y - b}{x - a}$. Rearranging this gives:



$$y - b = m(x - a)$$

NOTE: when you are asked to find the equation of a straight line, it is fine to leave it in this form (unless you are specifically asked to remove the brackets).

Example 2: Find the equations of the lines:

a) through $(4, 5)$ with $m = 2$

b) joining $(-1, -2)$ and $(3, 10)$

c) parallel to the line $x - 2y + 4 = 0$ and passing through the point $(2, -3)$

Example 3: Find the equation of the line through (-5, -1) with $m = \frac{2}{3}$, giving your answer in the form $Ax + By + C = 0$.

Example 4: Sketch the line $5x - 2y - 24 = 0$ by finding the points where it crosses the x – and y – axes.

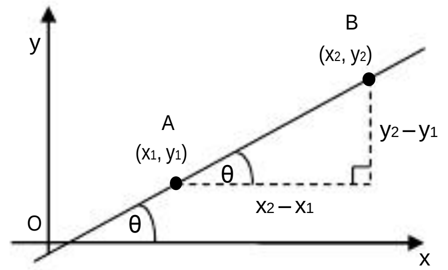
The Angle with the x-axis

The gradient of a line can also be described as the angle it makes with the positive direction of the x –axis.

As the y-difference is OPPOSITE the angle and the x-difference is ADJACENT to it, we get:

$m_{AB} \quad \tan\theta$

(where θ is measured CLOCKWISE from the x-axis)



Example 5: Find the angle made with the positive direction of the x -axis and the lines:

a) $y = x - 1$

b) $y = 5 - 3x$

c) joining the points (3, -2) and (7, 4)

Gradients of staight lines can be summarised as follows:

[redacted] direction of the x-axis
b) lines sloping down from left to right have negative gradients and make obtuse angles with the positive direction of the x-axis
c) lines with equal gradients are parallel
d) horizontal lines (parallel to the x -axis) have gradient zero and equation $y = a$
e) vertical lines (parallel to the y -axis) have gradient undefined and equation $x = b$

Collinearity

If three (or more) points lie on the same line, they are said to be collinear.

Example 6: Prove that the points D (-1, 5), E (0, 2) and F (4, -10) are collinear.

Perpendicular Lines

If two lines are perpendicular to each other (i.e. they meet at 90°), then:

$m_1 m_2 = -1$

Example 7: Show whether these pairs of lines are perpendicular:

a) $x+y+5=0$
$x - y - 7 = 0$

b) $2x - 3y = 5$
$3x = 2y + 9$

c) $y = 2x - 5$
$6y = 10 - 3x$

When asked to find the gradient of a line perpendicular to another, follow these steps:

1. Find the gradient of the given line
2. Flip it upside down
3. Change the sign (e.g. negative to positive)

Example 8: Find the gradients of the lines perpendicular to:

a) the line $y = 3x - 12$

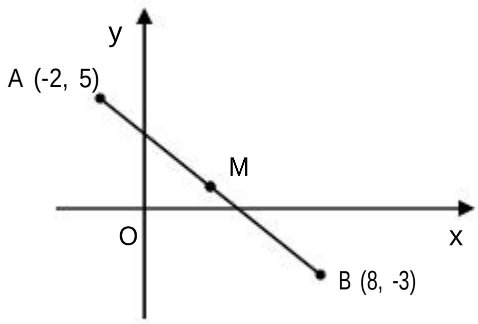
b) a line with gradient = -1.5

c) the line $2y + 5x = 0$

Example 9: Line L has equation $x + 4y + 2 = 0$. Find the equation of the line perpendicular to L which passes through the point (-2, 5).

Midpoints and Perpendicular Bisectors

The midpoint of a line lies exactly halfway along it. To find the coordinates of a midpoint, find halfway between the x – and y – coordinates of the points at each end of the line (see diagram).



The x – coordinate of M is halfway between -2 and 8, and its y – coordinate is halfway between 5 and -3.

In general, if M is the midpoint of A (x_1, y_1) and B (x_2, y_2):

$$M \quad \frac{x_1 \quad x_2}{2}, \frac{y_1 \quad y_2}{2}$$

The perpendicular bisector of a line passes through its midpoint at 90°.

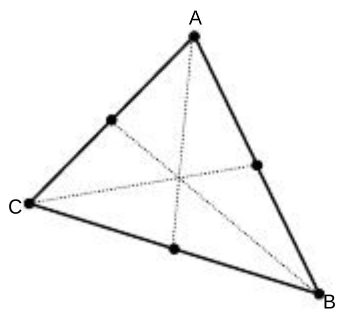
Example 10: Find the perpendicular bisector of the line joining F (-4, 2) and G (6, 8).

To find the equation of a perpendicular bisector:

- Find the gradient of the line joining the given points
- Find the perpendicular gradient (flip and make negative)
- Find the coordinates of the midpoint
- Substitute into $y - b = m(x - a)$

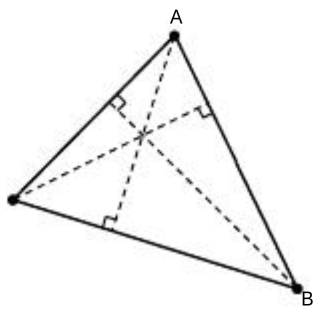
Lines Inside Triangles:
Medians, Altitudes & Perpendicular Bisectors

In a triangle, a line joining a corner to the midpoint of the [redacted]



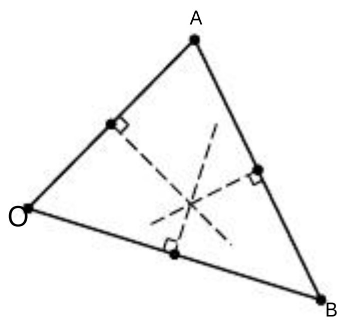
The medians are concurrent (i.e. meet at the same point) at the centoid, which divides each median in the ratio 2:1 The median always divides the area of a triangle in half. A solid triangle of uniform density will balance on the centroid.

A line through a corner which is perpendicular to the opposite side is called an altitude.



The altitudes are concurrent at the orthocentre. The orthocentre isn't always located inside the triangle e.g. if the triangle is obtuse.

A line at 90 to the midpoint is [redacted]

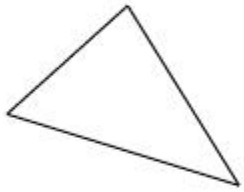


The perpendicular bisectors are concurrent at the circumcentre. The circumcentre is the centre of the circle touched by the vertices of the triangle.

For all triangles, the centroid, orthocentre and circumcentre are collinear.

Example 11: A triangle has vertices P (0, 2), Q (4, 4) and R (8, -6).

a) Find the equation of the median through P.



> To find the equation of a median:
>
> Find the midpoint of the side opposite the given point
> Find the gradient of the line joining the given point and the midpoint
> Substitute into $y - b = m(x - a)$

b) Find the equation of the altitude through R.

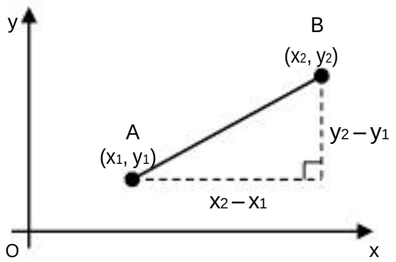
> To find the equation of an altitude:
>
> Find the gradient of the side opposite the given point
> Find the perpendicular gradient (flip and make negative)
> Substitute into $y - b = m(x - a)$

Distance between Two Points

The distance between any two points A (x_1, y_1) and B (x_2, y_2) can be found easily by Pythagoras' Theorem.

If d is the distance between A and B, then:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



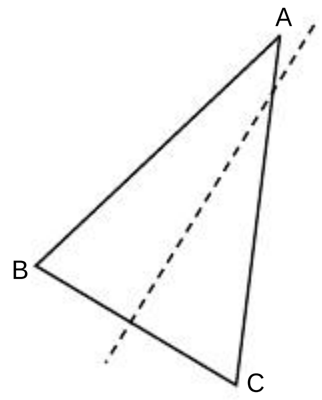
Example 12: Calculate the distance between:

a) A (-4, 4) and B (2, -4)

b) X (11, 2) and Y (-2, -5)

Example 13: A is the point (2, -1), B is (5, -2) and C is (7, 4). Show that BC = 2AB.

Past Paper Example 1: The vertices of triangle ABC are A(7, 9), B(-3, -1) and C(5, -5) as shown:
The broken line represents the perpendicular bisector of BC



a) Show that the equation of the perpendicular bisector of BC is $y = 2x - 5$

b) Find the equation of the median from C

c) Find the co-ordinates of the point of intersection of the perpendicular bisector of BC and the median from C.

Past Paper Example 2:

The line GH makes an angle of 30° with the y-axis as shown in the diagram opposite.

What is the gradient of GH?