SQA HIGHER. JANUARY 2002 – PAPER 1

1 (a) $A(-1,5) \quad B(3,1)$

$$m_{AB} = \frac{5-1}{-1-3} = \frac{4}{-4} = -1$$

Eqn. is $y-5 = -1(x+1)$

$$y - 5 = -x - 1$$

$$x + y - 4 = 0$$

(b) $m = \tan\theta = -1$

$$\therefore \theta = 135^\circ$$

2 (a) $\underline{u} + 3\underline{v} = \begin{pmatrix}1\\7\\-2\end{pmatrix} + \begin{pmatrix}3\\-6\\3\end{pmatrix}$

$$= \begin{pmatrix}4\\1\\1\end{pmatrix}$$

$$\underline{u} - 3\underline{v} = \begin{pmatrix}-2\\13\\-5\end{pmatrix}$$

Scalar ~~product~~ $= -8 + 13 - 5 = 0$

$\therefore \underline{u} + 3\underline{v}$ is perp. $\underline{u} - 3\underline{v}$

3 $y = 3x^{-1}$

$$y' = -3x^{-2} = -\frac{3}{x^2}$$

At P, $x = 1$, $y = 3$, $y' = -3$

$(1,3)$
$m = -3$

Eq. of ~~tangent~~ is

$$y - 3 = -3(x-1)$$

$$y - 3 = -3x + 3$$

$$3x + y - 6 = 0$$

4 (a) $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$\cos\frac{\pi}{3} = \frac{1}{2}.$$

(b) $\tan x = 4 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2}$

$$\tan x = \sqrt{3}$$

$$x = \frac{\pi}{3},\ \pi + \frac{\pi}{3}$$

$$x = \frac{\pi}{3},\ \frac{4\pi}{3}.$$

5 $f(x) = 3x^3 + 2x^2 + cx + d$

$f(2) = 24 + 8 + 2c + d = 0$

$2c + d = -32$ — ①

$f(-3) = -81 + 18 - 3c + d = 0$

$-3c + d = 63$ — ②

① − ②: $5c = -95$

$$c = -19$$

$\therefore d = -32 - 2c$

$$= -32 + 38$$

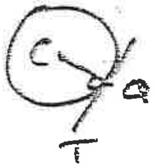
$$= 6$$

$$\boxed{c = -19,\ d = 6}$$

6 (a) $x^2+y^2+4x-10y+9=0$

Centre is $C(-2, 5)$

$Q(2, 3)$



$$m_{CQ} = \frac{5-3}{-2-2} = \frac{2}{-4} = -\frac{1}{2}$$

$\therefore m_{QT} = +2$

Eq. of QT is $y-3 = 2(x-2)$

$y-3 = 2x-4$

$y = 2x-1$

(b) $\left.\begin{array}{l} y = 2x-1 \\ y = -x^2+6x-5 \end{array}\right\}$

meet when

$2x-1 = -x^2+6x-5$

$x^2-4x+4 = 0$

$(x-2)(x-2) = 0$

$\left.\begin{array}{l} x = 2 \text{ (TWICE)} \\ y = 4-1 = 3 \end{array}\right\}$

$y = 2x-1$ is a tangent to the parabola at $(2,3)$

OR $\left.\begin{array}{l} y = -x^2+6x-5 \\ y' = -2x+6 \end{array}\right.$

At Q, $x = 2$, $y = -4+12-5 = 3$

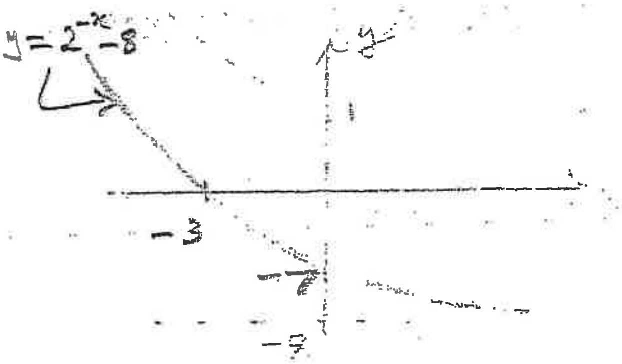
Tangent is $y-3 = 2(x-2)$

$y = 2x-1$

7 $\int x^{1/3} - x^{-1/2}\,dx$

$= \frac{3}{4}x^{4/3} - \frac{2}{1}x^{1/2} + C$

8 (a) Reflect in y-axis and move down 8 units.



$y = 2^{-x} - 8 = \frac{1}{2^x} - 8$

Cuts y-axis: $x = 0$, $y = \frac{1}{1} - 8 = -7$

$\boxed{(0, -7)}$

Cuts x-axis: $y = 0$

$\therefore \frac{1}{2^x} - 8 = 0$

$\frac{1}{2^x} = \frac{8}{1}$, $2^x = \frac{1}{8} = 2^{-3}$

$\therefore x = -3$

$\boxed{(-3, 0)}$

(53)

(9)

(a) $f(x) = \dfrac{3}{x+1}$

$$h(x) = f(f(x))$$
$$= f\left(\frac{3}{x+1}\right)$$
$$= \frac{3}{\frac{3}{x+1} + 1}$$
$$= \frac{3(x+1)}{(x+1)\left[\frac{3}{x+1} + 1\right]}$$
$$= \frac{3(x+1)}{3 + x + 1}$$
$$= \frac{3x+3}{x+4}$$

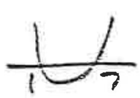
(b) Domain $\{x : x \neq -4\}$

(10)

$$f(x) = 2x + 3 + 18(x-4)^{-1}$$
$$f'(x) = 2 - 18(x-4)^{-2}$$
$$= 2 - \frac{18}{(x-4)^2}$$
$$= \frac{2(x-4)^2 - 18}{(x-4)^2}$$
$$= \frac{2[(x-4)^2 - 3^2]}{(x-4)^2}$$
$$= \frac{2(x-7)(x-1)}{(x-4)^2}$$

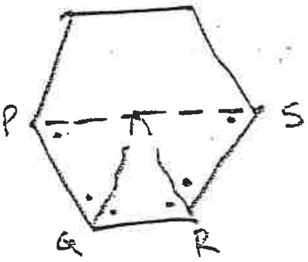
$f'(x) > 0$ when

$x < 1$ or $x > 7$



(11) $\overrightarrow{QS} = \underline{b} + \underline{c}$

• $60°$ angles



$Q\hat{R}S = 120°$

$\therefore R\hat{Q}S = \dfrac{180 - 120}{2} = 30°$

(*) $P\hat{Q}S = 120° - 30° = 90°$

$\therefore \underline{a} \cdot (\underline{b} + \underline{c}) = 0$

or $\underline{a} \cdot (\underline{b} + \underline{c})$

$$= \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$$
$$= |\underline{a}||\underline{b}| \cos 60° + |\underline{a}||\underline{c}| \cos 120°$$
$$= 4 \cos 60° + 4(-\cos 60°)$$
$$= 0$$

(12) $\log p + \log t = \cos^2 x + \sin^2 x$

$$= 1$$

$\therefore \log_a(pt) = 1$

$\therefore pt = a^1 = a$

Q11 (*) or:

CIRCLE WITH DIAMETER PS passes through all vertices

$\therefore P\hat{Q}S = 90°$

5

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14

1 Diagonals of rhombus are perpendicular.

(a)(i)

$m_{PR} = 2$

$\therefore m_{QS} = -\frac{1}{2}$

Eq of QS is $y - 4 = -\frac{1}{2}(x+2)$

$2y - 8 = -x - 2$

$x + 2y - 6 = 0$

(ii) At T, $\left.\begin{aligned} 2x - y - 2 &= 0 \\ x + 2y - 6 &= 0 \end{aligned}\right\}$

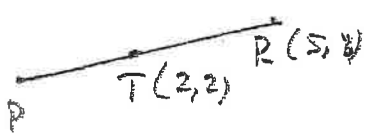
$\left.\begin{aligned} 4x - 2y - 4 &= 0 \\ x + 2y - 6 &= 0 \end{aligned}\right\}$

$5x - 10 = 0$

$x = 2,\ y = 4 - 2 = 2$

$\therefore$ T is $(2, 2)$

(b) T is mid point of RP



Hence P is $(-1, -4)$

2 $\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} -2 \\ 1 \\ 8 \end{pmatrix} - \begin{pmatrix} -8 \\ 10 \\ 20 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ -12 \end{pmatrix}$

$= 3\begin{pmatrix} 2 \\ -3 \\ -4 \end{pmatrix}$

$\overrightarrow{BC} = \underline{c} - \underline{b} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -4 \end{pmatrix}$

$\therefore \overrightarrow{AB} = 3\,\overrightarrow{BC}$

AB parallel to BC, B common

$\therefore$ A, B, C collinear. AB : BC = 3 : 1

3 (a) $u_{n+1} = 0.9u_n + 10$

Limit L

$L = 0.9 L + 10$

$\therefore 0.1 L = 10$

$L = \frac{10}{0.1} = 100$

(b) Half of limit $= 50$

$u_0 = 1$

$u_1 = 10.9$

$u_2 = 19.81$

$u_3 = 27.829$

$u_4 = 35.0461$

$u_5 = 41.54149$

$u_6 = 47.387341$

$u_7 = 52.6486069$

Least value of n is 7

$u_7 \doteq 52.6486$

4 (a) $\sqrt{3}\sin x^\circ + 1\cos x^\circ$

$= k\sin(x+a)^\circ$

$= k\sin x^\circ\cos a^\circ + k\cos x^\circ\sin a^\circ$

$\therefore \left.\begin{aligned} k\cos a^\circ &= \sqrt{3} \\ k\sin a^\circ &= 1 \end{aligned}\right\}$ $k = \sqrt{3+1} = 2$

$\tan a^\circ = \frac{1}{\sqrt{3}}$, $a = 30^\circ$

$\boxed{2\sin(x+30)^\circ}$

Max value of $5 + 2\sin(x+30)^\circ$ is 7 when $\sin(x+30)^\circ = 1$

$x + 30 = 90$

$x = 60$

(5) $\cos 2x - 2\sin^2 x = 0, \quad 0 \le x < 2\pi$

$1 - 2\sin^2 x - 2\sin^2 x = 0$

$1 - 4\sin^2 x = 0$

$4\sin^2 x = 1$

$\sin^2 x = \frac{1}{4}$

$\sin x = \frac{1}{2}$ or $\sin x = -\frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

(6) $f(x) = 2x^3 - 5x^2 - 3x + 1$

$f(0.1) = 2(0.001) - 5(0.01) - 3(0.1) + 1$
$= 0.002 - 0.05 - 0.3 + 1$
> 0

$f(0.2) = 2(0.008) - 5(0.04) - 3(0.2) + 1$
$= 0.016 - 0.20 - 0.6 + 1$
> 0

$f(0.3) = 2(0.027) - 5(0.09) - 3(0.3) + 1$
$= 0.054 - 0.45 - 0.9 + 1$
< 0

[f changes sign & so there is a root between 0.2 and 0.3]

$f(0.25) = -0.03125$
< 0

$\therefore f$ changes sign between 0.2 and 0.25

$\therefore x = 0.2$ (to 1 d.p.)

(7) $S = 1.7\,w(400 - w^2)$

$S(w) = 1.7(400w - w^3)$

$S'(w) = 1.7(400 - 3w^2)$

For stationary values, $S'(w) = 0$

$400 - 3w^2 = 0$

$w^2 = \frac{400}{3}$

$w = \frac{20}{\sqrt{3}} \quad (w > 0)$

≈ 11.54

w	10	11.54	12
S'	+	0	−
GRAPH	↗	→	↘

$\therefore S$ has a max when $w = \frac{20}{\sqrt{3}}$

$w^2 + d^2 = 20^2$

$\therefore d^2 = 20^2 - w^2 = 400 - \frac{400}{3}$

$= 400\left(1 - \frac{1}{3}\right) = \frac{800}{3}$

$\therefore d = \sqrt{\frac{800}{3}} = 20\sqrt{\frac{2}{3}}$

$= \frac{20\sqrt{2}.\sqrt{3}}{3} = \frac{20\sqrt{6}}{3}$

Dimensions are

$w = \frac{20}{\sqrt{3}}, \quad d = \frac{20\sqrt{6}}{3}$

$= \frac{20\sqrt{3}}{3}$

(8)

$$\int_0^1 \cos 3x - \sin\left(\tfrac{1}{3}x+1\right) dx$$

$$= \left[\tfrac{1}{3}\sin 3x + 3\cos\left(\tfrac{1}{3}x+1\right)\right]_0^1$$

$$= \tfrac{1}{3}\sin 3 + 3\cos\tfrac{4}{3} - \left(\tfrac{1}{3}\sin 0 + 3\cos 1\right)$$

$$= 0.04704 + 0.70571 - 1.62090$$

$$\doteq -0.868 \text{ (to 3 d.p.)}$$

(9) $N(t) = 950\,(2.6)^{0.2t}$

(a) $t=0$, $N(0) = 950$

(b) $N = 9500$

$$950\,(2.6)^{0.2t} = 9500$$

$$(2.6)^{0.2t} = 10$$

$$\log\left[(2.6)^{0.2t}\right] = \log 10$$

$$0.2t \log_{10} 2.6 = \log_{10} 10 = 1$$

$$\therefore t = \frac{1}{0.2 \log_{10} 2.6}$$

$$= 12.0489$$

TIME TAKEN = 12.0489 hours

$\approx$ 12 hr. 3 min.

(10) $y + 2x = k$

$$y = k - 2x$$

Meets $x^2+y^2-2x-4=0$

when

$$x^2+(k-2x)^2-2x-4=0$$

$$x^2+k^2-4kx+4x^2-2x-4=0$$

$$5x^2 - 2(2k+1)x + (k^2-4)=0$$

For equal roots,

$$b^2 - 4ac = 0$$

$$\cancel{4}(2k+1)^2 - \cancel{4}\cdot 5\cdot(k^2-4)=0$$

$$4k^2+4k+1-5k^2+20=0$$

$$-k^2+4k+21=0$$

$$k^2-4k-21=0$$

$$(k+3)(k-7)=0$$

$$k=-3 \text{ or } k=7$$

But $k>0$ $\therefore \boxed{k=7}$

When $k=7$, quad. eqn is:

$$5x^2-2(14+1)x+(49-4)=0$$

$$5x^2-30x+45=0$$

$$x^2-6x+9=0$$

$$(x-3)(x-3)=0$$

$x=3$ TWICE

$$y = k-2x = 7-6 = 1$$

$\therefore$ POINT OF CONTACT IS $(3, 1)$

(11) 1^{st} AREA

$= \int_{-5}^{0} f(x) - (-6) \; dx$

$= \int -\frac{1}{4}x^2 - \frac{5}{4}x + 6 \; dx$

$= \left[-\frac{1}{4} \cdot \frac{x^3}{3} - \frac{5}{4} \cdot \frac{x^2}{2} + 6x \right]_{-5}^{0}$

$= 0 - \left(\frac{125}{12} - \frac{125}{8} - 30 \right)$

$= 30 - \frac{125}{12} + \frac{125}{8}$

$= 30 + 125\left(\frac{1}{8} - \frac{1}{12} \right)$

$= 30 + 125 \cdot \frac{3 - 2}{24}$

$= 30 + \frac{125}{24}$

2^{nd} area

$= \int_{0}^{5} g(x) - (-6) \; dx$

$= \int \frac{1}{12}x^2 - \frac{5}{12}x + 6 \; dx$

$= \left[\frac{x^3}{36} - \frac{5x^2}{24} + 6x \right]_{0}^{5}$

$= \frac{125}{36} - \frac{125}{24} + 30$

$= 30 + \frac{125}{12}\left(\frac{1}{3} - \frac{1}{2} \right)$

$= 30 + \frac{125}{12} \cdot \frac{2-3}{6}$

$= 30 - \frac{125}{72}$

TOTAL AREA

$= 30 + \frac{125}{24} + 30 - \frac{125}{72}$

$= 60 + 125\left(\frac{1}{24} - \frac{1}{72} \right)$

$= 60 + 125 \; \frac{3 - 1}{72}$

$= 60 + \frac{125}{36}$

$= 60 + 3 + \frac{17}{36}$

$= 63 \frac{17}{36}$

(b) Max energy

$= 63\frac{17}{36} \div 10$

≈ 6.347

$= 6$ kW