SCOTTISH CERTIFICATE OF EDUCATION

MATHEMATICS (REVISED)

Higher Grade—PAPER I

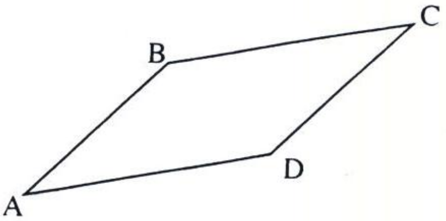
All questions should be attempted

Marks

1. Find $\int(3x^3 + 4x)dx$. **(3)**

2. If $f(x) = kx^3 + 5x - 1$ and $f'(1) = 14$, find the value of k. **(3)**

3. A is the point $(2, -1, 4)$, B is $(7, 1, 3)$ and C is $(-6, 4, 2)$. If ABCD is a parallelogram, find the coordinates of D.



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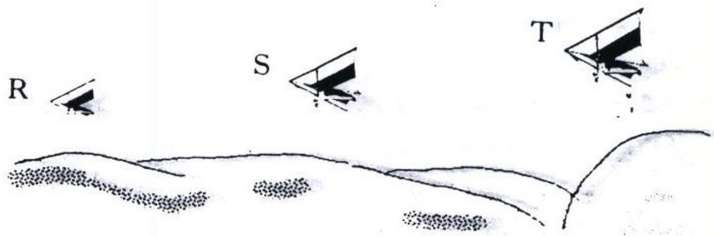
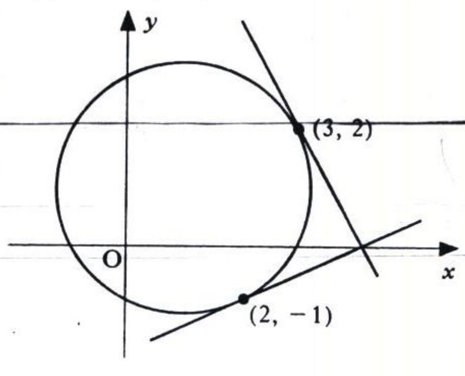


Relative to the top of a hill, three gliders have positions given by $R(-1, -8, -2)$, $S(2, -5, 4)$ and $T(3, -4, 6)$.
Prove that R, S and T are collinear. **(3)**

5. The circle shown in the diagram has equation $(x - 1)^2 + (y - 1)^2 = 5$.

Tangents are drawn at the points $(3, 2)$ and $(2, -1)$.

Write down the coordinates of the centre of the circle and hence show that the tangents are perpendicular to each other.



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6. Find algebraically the **exact** value of $\sin\theta^\circ + \sin(\theta + 120)^\circ + \cos(\theta + 150)^\circ$. **(3)**

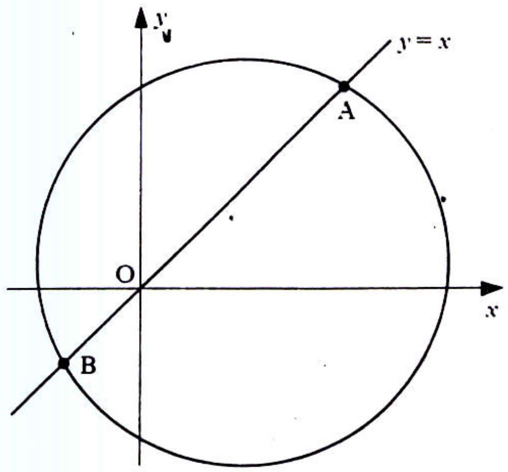
7. If $\boldsymbol{u} = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$, write down the components of $\boldsymbol{u} + \boldsymbol{v}$ and $\boldsymbol{u} - \boldsymbol{v}$.

Hence show that $\boldsymbol{u} + \boldsymbol{v}$ and $\boldsymbol{u} - \boldsymbol{v}$ are perpendicular. **(3)**

8. The straight line $y = x$ cuts the circle $x^2 + y^2 - 6x - 2y - 24 = 0$ at A and B.

(*a*) Find the coordinates of A and B. **(3)**

(*b*) Find the equation of the circle which has AB as diameter. **(3)**



9. A sequence is defined by the recurrence relation

$$u_n = 0{\cdot}9u_{n-1} + 2, \quad u_1 = 3.$$

(*a*) Calculate the value of u_2. **(1)**

(*b*) What is the smallest value of n for which $u_n > 10$? **(1)**

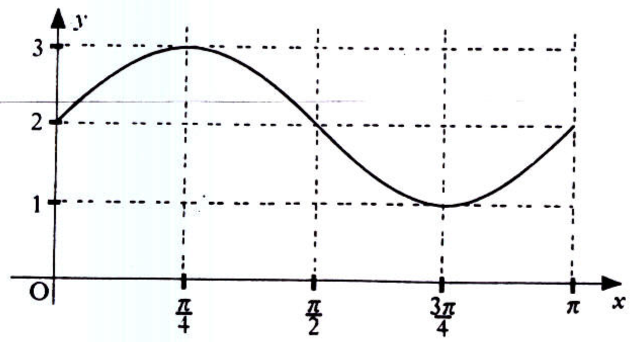
(*c*) Find the limit of this sequence as $n \to \infty$. **(2)**

10. Find the derivative, with respect to x, of $\frac{1}{x^3} + \cos 3x$. **(4)**

11. Show that $x^2 + 8x + 18$ can be written in the form $(x + a)^2 + b$.

Hence or otherwise find the coordinates of the turning point of the curve with equation $y = x^2 + 8x + 18$. **(3)**

12. The diagram shows the graph of the function $y = a + b\sin cx$ for $0 \leq x \leq \pi$.



(*a*) Write down the values of a, b and c. **(3)**

(*b*) Find algebraically the values of x for which $y = 2{\cdot}5$. **(3)**

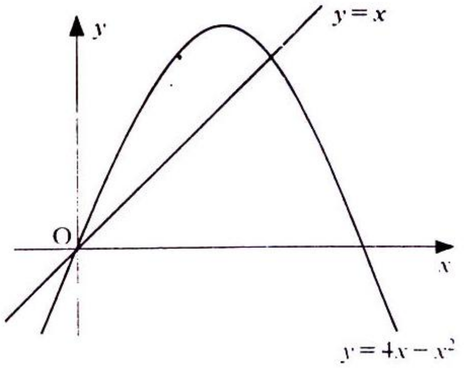
13. If $\cos\theta = \frac{4}{5}$, $0 \le \theta < \frac{\pi}{2}$, find the **exact** value of

(*a*) $\sin 2\theta$ **(2)**

(*b*) $\sin 4\theta$. **(3)**

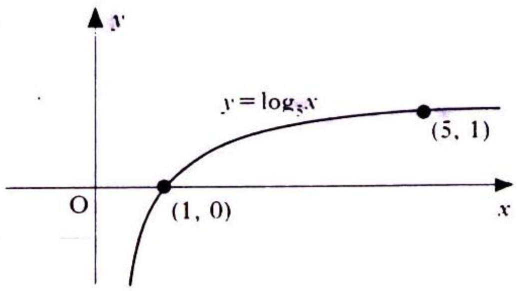
14. Find the gradient of the tangent to the parabola $y = 4x - x^2$ at (0, 0).

Hence calculate the size of the angle between the line $y = x$ and this tangent. **(6)**



15. Solve algebraically the equation $\cos 2x^\circ + 5\cos x^\circ - 2 = 0$. $0 \le x < 360$. **(5)**

16.



The diagram shows a sketch of part of the graph of $y = \log_5 x$.

(*a*) Make a copy of the graph of $y = \log_5 x$.

On your copy, sketch the graph of $y = \log_5 x + 1$.

Find the coordinates of the point where it crosses the x-axis. **(3)**

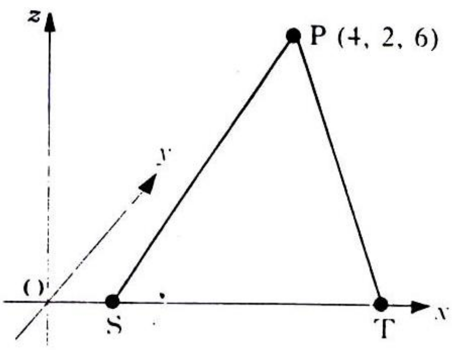
(*b*) Make a second copy of the graph of $y = \log_5 x$.

On your copy, sketch the graph of $y = \log_5 \frac{1}{x}$. **(2)**

17. Differentiate $\sin^3 x$ with respect to x.

Hence find $\int \sin^2 x \cos x \, dx$. **(4)**

18. The diagram shows a point P with coordinates (4, 2, 6) and two points S and T which lie on the x-axis. If P is 7 units from S and 7 units from T, find the coordinates of S and T.



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19. A function f is defined on the set of real numbers by

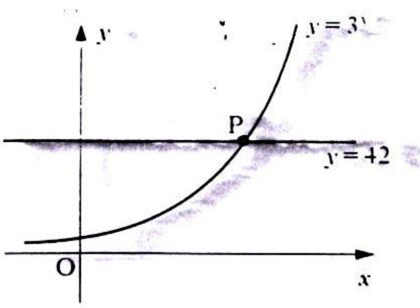
$$f(x) = \frac{x}{1 - x}, \quad (x \neq 1).$$

Find, in its simplest form, an expression for $f(f(x))$.

(3)

20. The diagram shows part of the graph with equation $y = 3^x$ and the straight line with equation $y = 42$. These graphs intersect at P.

Solve algebraically the equation $3^x = 42$, and hence write down, correct to 3 decimal places, the coordinates of P.



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SCOTTISH CERTIFICATE OF EDUCATION

MATHEMATICS (REVISED)

Higher Grade—PAPER II

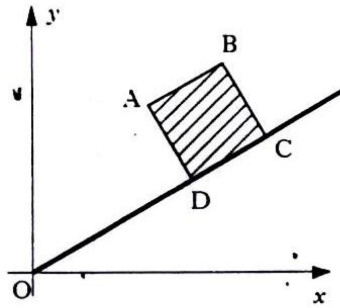
All questions should be attempted *Marks*

1. The graph of the curve with equation $y = 2x^3 + x^2 - 13x + a$ crosses the x-axis at the point (2, 0).

   (*a*) Find the value of a and hence write down the coordinates of the point at which this curve crosses the y-axis. **(3)**

   (*b*) Find algebraically the coordinates of the other points at which the curve crosses the x-axis. **(4)**

2. ABCD is a square. A is the point with coordinates (3, 4) and ODC has equation $y = \frac{1}{2}x$.

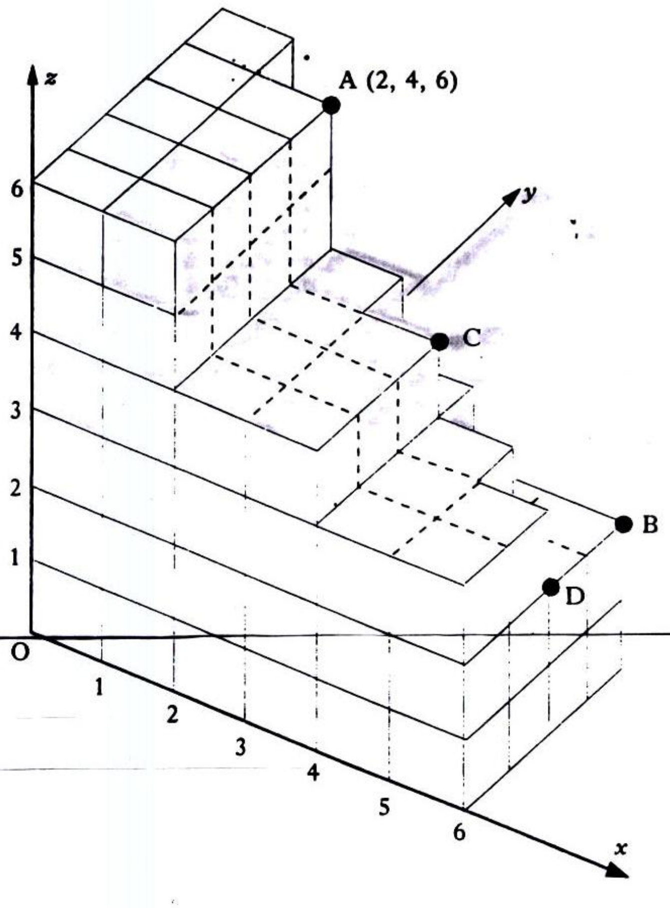


(*a*) Find the equation of the line AD. **(3)**

(*b*) Find the coordinates of D. **(3)**

(*c*) Find the area of the square ABCD. **(2)**

3.



With coordinate axes as shown, the point A is (2, 4, 6).

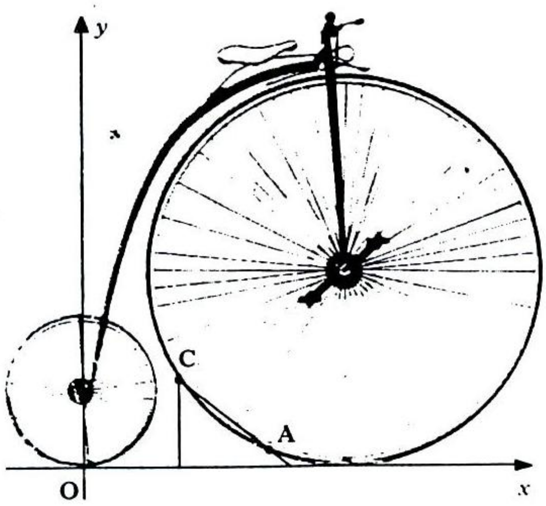
(*a*) Write down the coordinates of B, C and D. **(3)**

(*b*) Show that C is the midpoint of AD. **(1)**

(*c*) By using the components of the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$, calculate the size of angle AOB, where O is the origin. **(4)**

(*d*) Hence calculate the size of angle OAB. **(2)**

4.



A penny-farthing bicycle on display in a museum is supported by a stand at points A and C. A and C lie on the front wheel.

With coordinate axes as shown and 1 unit = 5cm,
the equation of the rear wheel (the small wheel) is $x^2 + y^2 - 6y = 0$ and the equation of the front wheel is $x^2 + y^2 - 28x - 20y + 196 = 0$.

(*a*) (i) Find the distance between the centres of the two wheels.

(ii) Hence calculate the clearance, ie the smallest gap, between the front and rear wheels. Give your answer to the nearest millimetre. **(8)**

(*b*) B(7, 3) is half-way between A and C, and P is the centre of the front wheel.

(i) Find the gradient of PB.

(ii) Hence find the equation of AC and the coordinates of A and C. **(8)**

5. (*a*) Express $3\sin x° - \cos x°$ in the form $k\sin(x - \alpha)°$, where $k > 0$ and $0 \leq \alpha \leq 90$. **(4)**

(*b*) Hence find algebraically the values of x between 0 and 180 for which $3\sin x° - \cos x° = \sqrt{5}$. **(4)**

(*c*) Find the range of values of x between 0 and 180 for which $3\sin x° - \cos x° \leq \sqrt{5}$. **(2)**

Marks

6. EXAMPLE

(i) Let $f(x) = x^3 + 5x - 1$.
Since $f(0) = -1$ and $f(1) = 5$,
the equation $f(x) = 0$ has a root in the interval $0 < x < 1$ because $f(0) < 0$ and $f(1) > 0$.

(ii) To find this root, the equation $x^3 + 5x - 1 = 0$ can be rearranged as follows:

$$x^3 + 5x - 1 = 0$$
$$x^3 + 5x = 1$$
$$x(x^2 + 5) = 1$$
$$x = \frac{1}{x^2 + 5}$$

We can write this result as a recurrence relation

$$x_{n-1} = \frac{1}{x_n^2 + 5}$$

and use it to find this root. In this example we will work to 3 decimal places and can therefore give the final answer to 2 decimal places.

(iii) For our first estimate, x_1, we use the mid-point of the interval $0 < x < 1$ [from part (i)].

$$x_1 = 0{\cdot}5, \qquad x_2 = \frac{1}{0{\cdot}5^2 + 5} = 0{\cdot}190$$

$$x_2 = 0{\cdot}190 \qquad x_3 = \frac{1}{0{\cdot}190^2 + 5} = 0{\cdot}199$$

$$x_3 = 0{\cdot}199 \qquad x_4 = \frac{1}{0{\cdot}199^2 + 5} = 0{\cdot}198$$

$$x_4 = 0{\cdot}198 \qquad x_5 = \frac{1}{0{\cdot}198^2 + 5} = 0{\cdot}198$$

Hence, correct to 2 decimal places, the root is $x = 0{\cdot}20$.

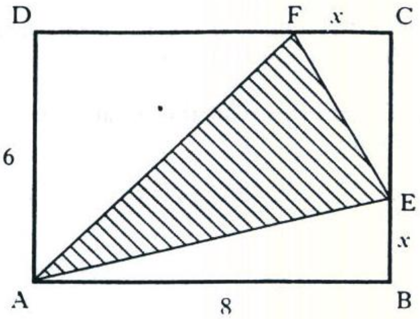
(*a*) Show that the equation $2x^3 + 3x - 1 = 0$ has a root in the interval $0 < x < 0{\cdot}5$. **(2)**

(*b*) By using the technique described above, find this root correct to 2 decimal places. **(6)**

7. A yacht club is designing its new flag.

The flag is to consist of a red triangle on a yellow rectangular background.

In the yellow rectangle ABCD, AB measures 8 units and AD is 6 units. E and F lie on BC and CD, x units from B and C as shown in the diagram.



(*a*) Show that the area, H square units, of the red triangle AEF is given by $H(x) = 24 - 4x + \frac{1}{2}x^2$. **(4)**

(*b*) Hence find the greatest and least possible values of the area of triangle AEF. **(8)**

8. (*a*) $f(x) = 4x^2 - 3x + 5$.

Show that $f(x+1)$ simplifies to $4x^2 + 5x + 6$ and find a similar expression for $f(x-1)$.

Hence show that $\dfrac{f(x+1) - f(x-1)}{2}$ simplifies to $8x - 3$. **(5)**

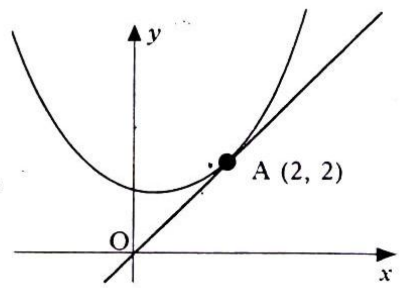
(*b*) $g(x) = 2x^2 + 7x - 8$.

Find a similar expression for $\dfrac{g(x+1) - g(x-1)}{2}$. **(4)**

(*c*) By examining your answers for (*a*) and (*b*), **write down** the simplified expression for $\dfrac{h(x+1) - h(x-1)}{2}$, where $h(x) = 3x^2 + 5x - 1$. **(2)**

9. (*a*) The point A(2, 2) lies on the parabola $y = x^2 + px + q$.

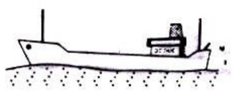
Find a relationship between p and q. **(1)**



(*b*) The tangent to the parabola at A is the line $y = x$. Find the value of p. Hence find the equation of the parabola. **(6)**

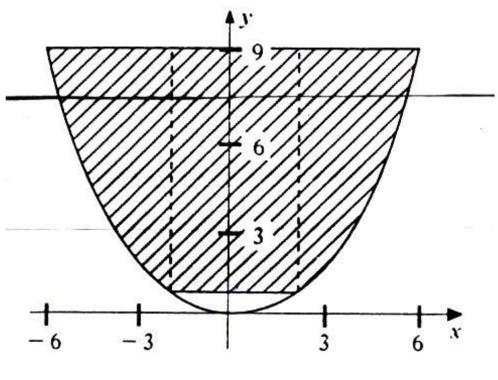
(*c*) Using your answers for p and q, find the value of the discriminant of $x^2 + px + q = 0$. What feature of the above sketch is confirmed by this value? **(2)**

10. The cargo space of a small bulk carrier is 60m long.



The shaded part of the diagram below represents the uniform cross-section of this space. It is shaped like the parabola with equation $y = \frac{1}{4}x^2$, $-6 \leq x \leq 6$, between the lines $y = 1$ and $y = 9$.

Find the area of this cross-section and hence find the volume of cargo that this ship can carry. **(9)**



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