SCOTTISH CERTIFICATE OF EDUCATION 1993

MONDAY, 10 MAY
9.30 AM – 11.30 AM

MATHEMATICS
HIGHER GRADE
Paper I

All questions should be attempted

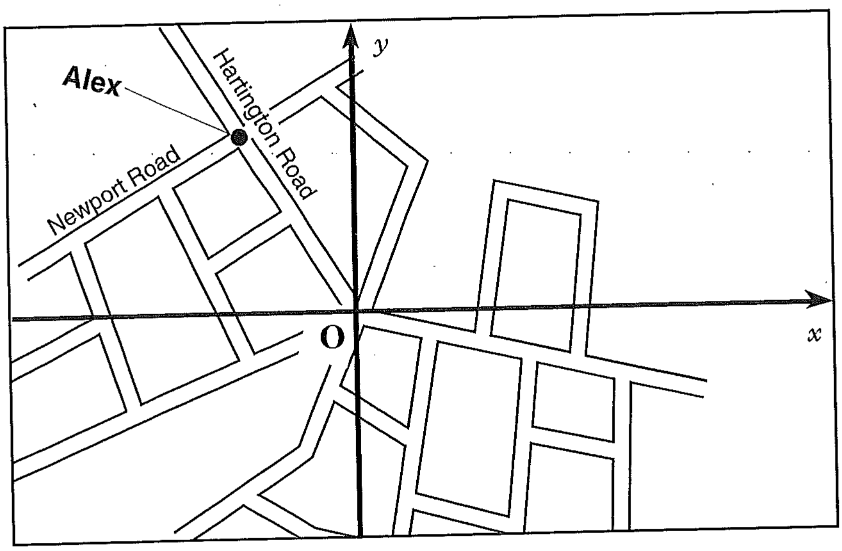
Marks

1. A is the point $(-3,2,4)$ and B is $(-1,3,2)$. Find

(*a*) the components of vector $\overrightarrow{AB}$; **(1)**

(*b*) the length of AB. **(2)**

2. Relative to the axes shown and with an appropriate scale, Alex stands at the point $(-2,3)$ where Hartington Road meets Newport Road.



(*a*) Find the equation of Newport Road which is perpendicular to Hartington Road. **(3)**

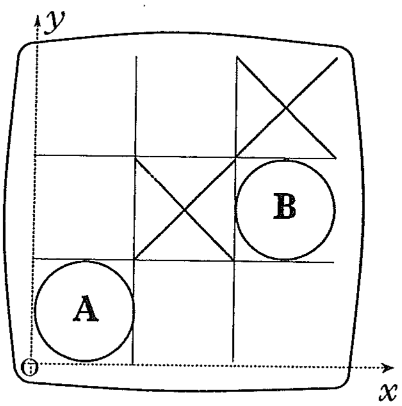
(*b*) Brenda is waiting for a bus at the point $(-5,1)$. Show that Brenda is standing on Newport Road. **(1)**

3. Find the values of k for which the equation $2x^2+4x+k=0$ has real roots. **(2)**

4. Find the x-coordinate of each of the points on the curve $y=2x^3-3x^2-12x+20$ at which the tangent is parallel to the x-axis. **(4)**

Marks

5. This diagram shows a computer-generated display of a game of noughts and crosses. Relative to the coordinate axes which have been added to the display, the "nought" at A is represented by a circle with equation $(x-2)^2+(y-2)^2=4$.



(*a*) Find the centre of the circle at B. **(3)**

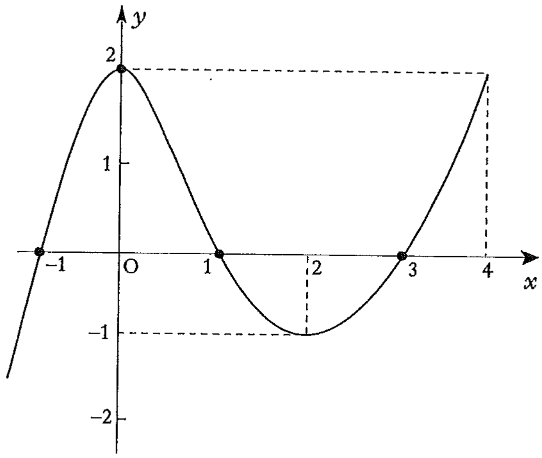
(*b*) Find the equation of the circle at B. **(1)**

6. For acute angles P and Q, $\sin P = \frac{12}{13}$ and $\sin Q = \frac{3}{5}$.

Show that the **exact** value of $\sin(P+Q)$ is $\frac{63}{65}$. **(3)**

7. One root of the equation $2x^3 - 3x^2 + px + 30 = 0$ is -3. Find the value of p and the other roots. **(4)**

8. The diagram shows the graph of $y = f(x)$.

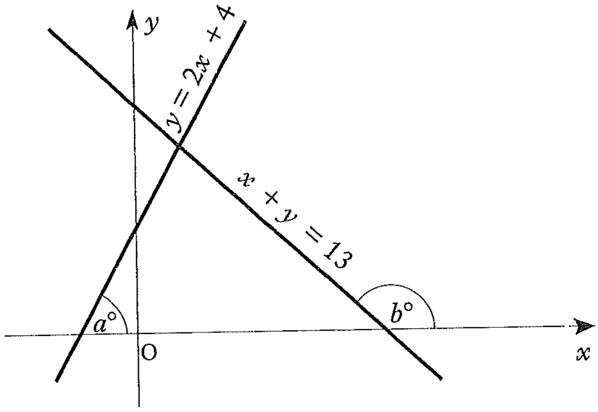


Sketch the graph of $y = 2 - f(x)$. **(3)**

9. Differentiate $4\sqrt{x} + 3\cos 2x$. **(4)**

Marks

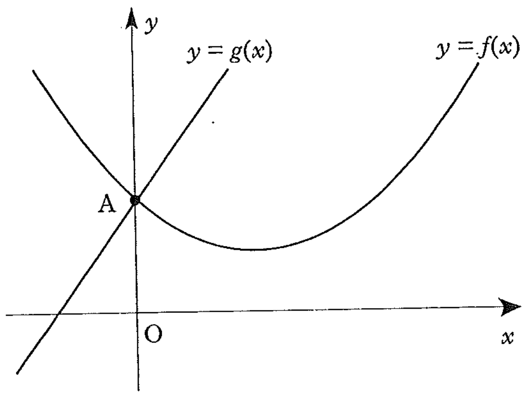
10. The lines $y = 2x + 4$ and $x + y = 13$ make angles of $a°$ and $b°$ with the positive direction of the x-axis, as shown in the diagram.



(*a*) Find the values of a and b. **(4)**

(*b*) **Hence** find the acute angle between the two given lines. **(1)**

11. The graphs of $y = f(x)$ and $y = g(x)$ intersect at the point A on the y-axis, as shown on the diagram.



If $g(x) = 3x + 4$
and $f'(x) = 2x - 3$, find $f(x)$. **(4)**

12. The vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are defined as follows:

$$\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{k}, \quad \boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}, \quad \boldsymbol{c} = -\boldsymbol{j} + \boldsymbol{k}.$$

(*a*) Evaluate $\boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c}$. **(3)**

(*b*) From your answer to part (*a*), make a deduction about the vector $\boldsymbol{b} + \boldsymbol{c}$. **(2)**

Marks

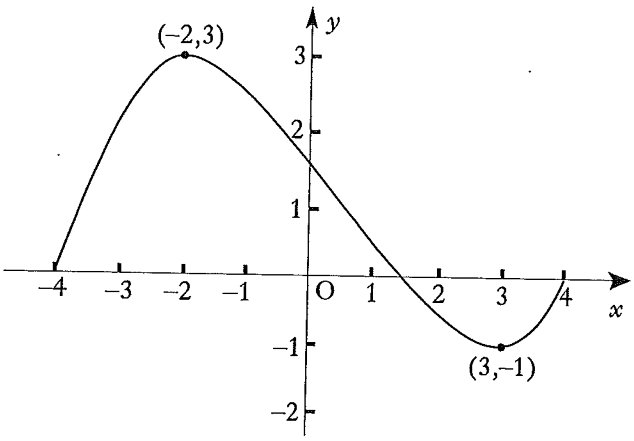
13. $f(x) = 2x - 1$, $g(x) = 3 - 2x$ and $h(x) = \frac{1}{4}(5 - x)$.

(*a*) Find a formula for $k(x)$ where $k(x) = f(g(x))$. **(2)**

(*b*) Find a formula for $h(k(x))$. **(2)**

(*c*) What is the connection between the functions h and k? **(1)**

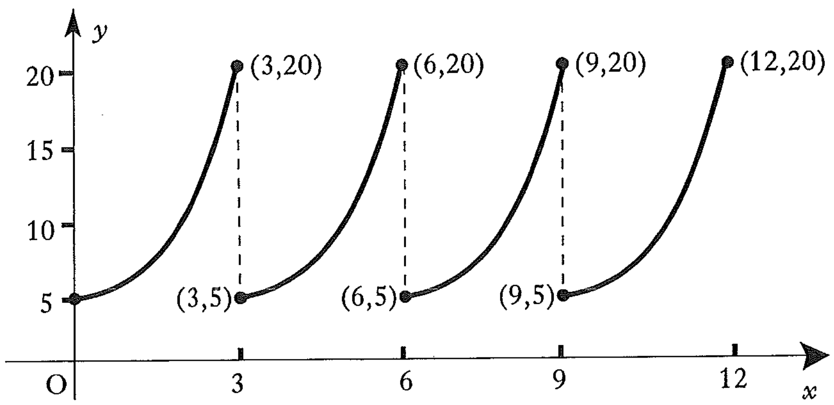
14. A sketch of a cubic function, f, with domain $-4 \le x \le 4$, is shown in the diagram below.



Sketch the graph of the derived function, f', for the same domain. **(3)**

Marks

15. A medical technician obtains this print-out of a wave form generated by an oscilloscope.



The technician knows that the equation of the first branch of the graph (for $0 \leq x \leq 3$) should be of the form $y = ae^{kx}$.

(a) Find the values of a and k. **(4)**

(b) Find the equation of the second branch of the curve (i.e. for $3 \leq x \leq 6$). **(1)**

16. Find $\int \sqrt{1+3x}\,dx$ and hence find the **exact** value of $\int_0^1 \sqrt{1+3x}\,dx$ **(4)**

17. If $f(a) = 6\sin^2 a - \cos a$, express $f(a)$ in the form $p\cos^2 a + q\cos a + r$.

Hence solve, correct to three decimal places, the equation

$$6\sin^2 a - \cos a = 5 \text{ for } 0 \leq a \leq \pi.$$ **(4)**

Marks

18. Explain why the equation $x^2+y^2+2x+3y+5=0$ does **not** represent a circle. **(2)**

19. (*a*) Show that $(\cos x+\sin x)^2=1+\sin 2x$. **(1)**

(*b*) Hence find $\int (\cos x+\sin x)^2\,dx$. **(3)**

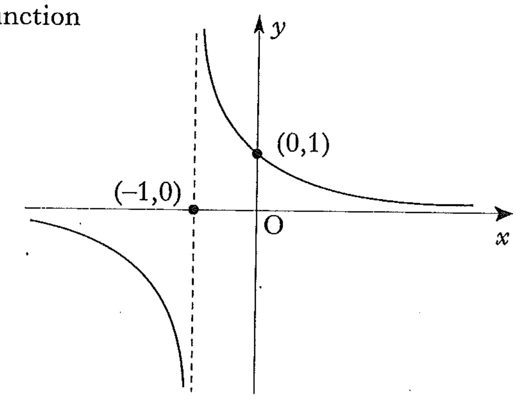
20. The point P (p,k) lies on the curve with equation $y=\log_e x$.

The point Q (q,k) lies on the curve with equation $y=\frac{1}{2}\log_e x$.

Find a relationship between p and q and hence find q when $p=5$. **(4)**

21. The diagram shows the graph of the function

$$f(x)=\frac{1}{x+1},\quad x\neq -1.$$



Prove that the function f is decreasing for all values of x except $x=-1$. **(4)**

[END OF QUESTION PAPER]

SCOTTISH CERTIFICATE OF EDUCATION 1993

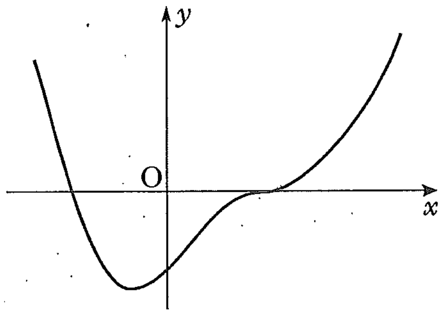
MONDAY, 10 MAY
1.30 PM – 4.00 PM

MATHEMATICS
HIGHER GRADE
Paper II

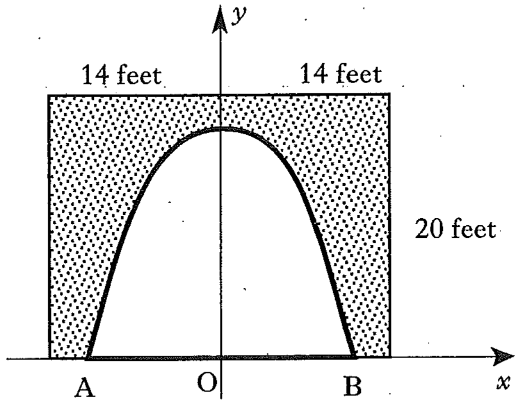
All questions should be attempted

Marks

1. The function f, whose incomplete graph is shown in the diagram, is defined by $f(x) = x^4 - 2x^3 + 2x - 1$. Find the coordinates of the stationary points and justify their nature. **(8)**



2. The concrete on the 20 feet by 28 feet rectangular facing of the entrance to an underground cavern is to be repainted.

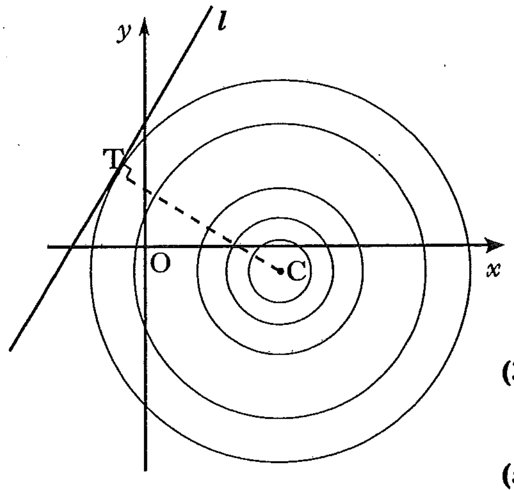


Coordinate axes are chosen as shown in the diagram with a scale of 1 unit equal to 1 foot. The roof is in the form of a parabola with equation $y = 18 - \frac{1}{8}x^2$.

(*a*) Find the coordinates of the points A and B. **(2)**

(*b*) Calculate the total cost of repainting the facing at £3 per square foot. **(4)**

3. In an experiment with a ripple tank, a series of concentric circles with centre C(4,–1) is formed as shown in the diagram.
The line $\boldsymbol{l}$ with equation $y = 2x + 6$ represents a barrier placed in the tank.
The largest complete circle touches the barrier at the point T.



(*a*) Find the equation of the radius CT. **(3)**

(*b*) Find the equation of the largest complete circle. **(5)**

Marks

4. An array of numbers such as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called a matrix.

The eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are defined to be the roots of the equation $(a-x)(d-x)-bc=0$.

EXAMPLE

In order to find the eigenvalues of the matrix $\mathbf{B} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

solve $(1-x)(2-x)-4\times 3=0$

solution:

$$2-3x+x^2-12=0$$
$$x^2-3x-10=0$$
$$(x+2)(x-5)=0$$
$$x=-2 \text{ or } x=5$$

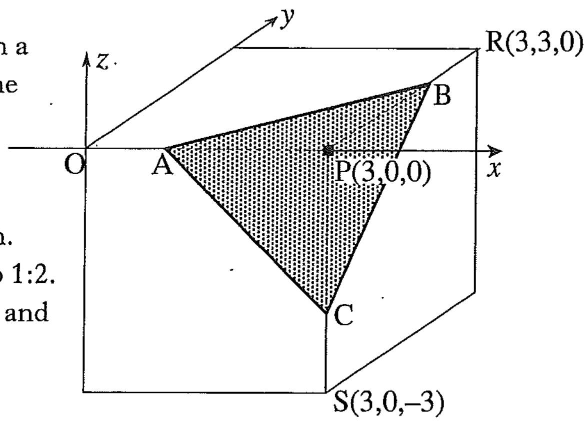
so the eigenvalues of **B** are −2 and 5

(*a*) Find the eigenvalues of $\mathbf{C} = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$. **(3)**

(*b*) Find the value of t for which the eigenvalues of the matrix $\mathbf{D} = \begin{pmatrix} 3 & -1 \\ t & 1 \end{pmatrix}$ are equal. **(5)**

Marks

5. A model of a crystal was made from a cube of side 3 units by slicing off the corner at P to leave a triangular face ABC.
Coordinate axes have been introduced as shown in the diagram.
The point A divides OP in the ratio 1:2.
Points B and C similarly divide RP and SP respectively in the ratio 1:2.

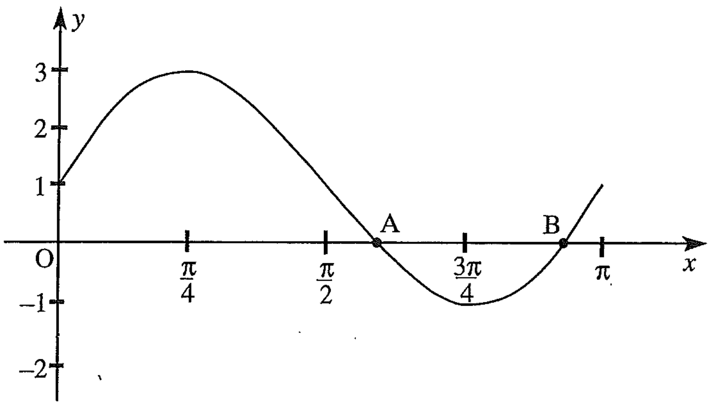


(*a*) Find the coordinates of A, B and C. **(3)**

(*b*) Calculate the area of triangle ABC. **(4)**

(*c*) Calculate the percentage increase or decrease in the surface area of the crystal compared with the cube. **(5)**

6. The diagram below shows the graph of $y = 2\sin 2x + 1$ for $0 \le x \le \pi$.



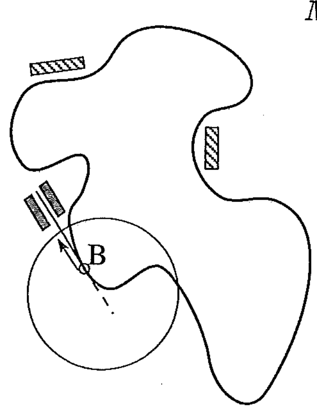
(*a*) Find the coordinates of A and B (as shown in the diagram) by solving an appropriate equation algebraically. **(5)**

(*b*) The points (0, 2) and (π, 0) are joined by a straight line ***l***. In how many points does ***l*** intersect the given graph? **(1)**

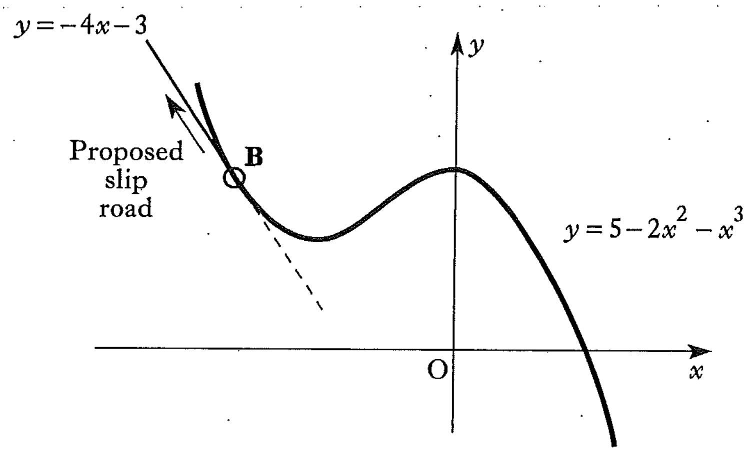
(*c*) C is the point on the given graph with an x-coordinate of $\frac{\pi}{2}$. Explain whether C is above, below or on the line ***l***. **(3)**

Marks

7. The diagram shows the plans for a proposed new racing circuit. The designer wishes to introduce a slip road at B for cars wishing to exit from the circuit to go into the pits. The designer needs to ensure that the two sections of road touch at B in order that drivers may drive straight on when they leave the circuit.



Relative to appropriate axes, the part of the circuit circled above is shown below. This part of the circuit is represented by a curve with equation $y = 5 - 2x^2 - x^3$ and the proposed slip road is represented by a straight line with equation $y = -4x - 3$.



(*a*) Find algebraically the coordinates of B. **(7)**

(*b*) Justify the designer's decision that this direction for the slip road does allow drivers to go straight on. **(1)**

Marks

8. Secret Agent 004 has been captured and his captors are giving him a 25 milligram dose of a truth serum every 4 hours. 15% of the truth serum present in his body is lost every hour.

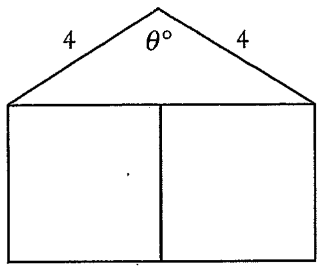
(*a*) Calculate how many milligrams of serum remain in his body after 4 hours (that is, immediately **before** the second dose is given). **(3)**

(*b*) It is known that the level of serum in the body has to be continuously above 20 milligrams before the victim starts to confess. Find how many doses are needed before the captors should begin their interrogation. **(3)**

(*c*) Let u_n be the amount of serum (in milligrams) in his body just **after** his n^{th} dose. Show that $u_{n+1} = 0 \cdot 522u_n + 25$. **(1)**

(*d*) It is also known that 55 milligrams of this serum in the body will prove fatal, and the captors wish to keep Agent 004 alive. Is there any maximum length of time for which they can continue to administer this serum and still keep him alive? **(4)**

9. A builder has obtained a large supply of 4 metre rafters. He wishes to use them to build some holiday chalets. The planning department insists that the gable end of each chalet should be in the form of an isosceles triangle surmounting two squares, as shown in the diagram.



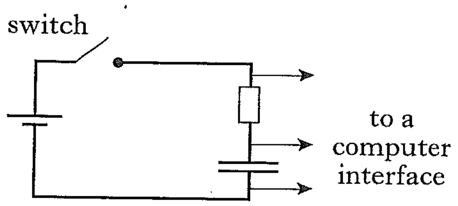
(*a*) If $\theta°$ is the angle shown in the diagram and A is the area (in square metres) of the gable end, show that

$$A = 8\left(2 + \sin\theta° - 2\cos\theta°\right).$$ **(5)**

(*b*) Express $8\sin\theta° - 16\cos\theta°$ in the form $k\sin(\theta - \alpha)°$. **(4)**

(*c*) Find algebraically the value of θ for which the area of the gable end is 30 square metres. **(4)**

10. When the switch in this circuit was closed, the computer printed out a graph of the current flowing (I microamps) against the time (t seconds). This graph is shown in figure 1. *Marks*



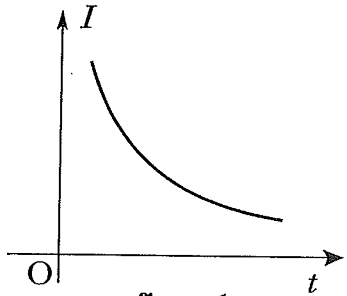


figure 1

In order to determine the equation of the graph shown in figure 1, values of $\log_e I$ were plotted against $\log_e t$ and the best fitting straight line was drawn as shown in figure 2.

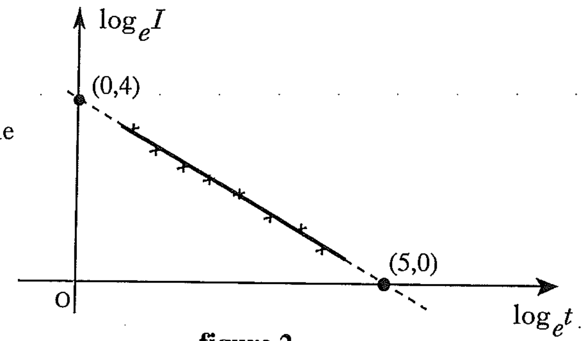


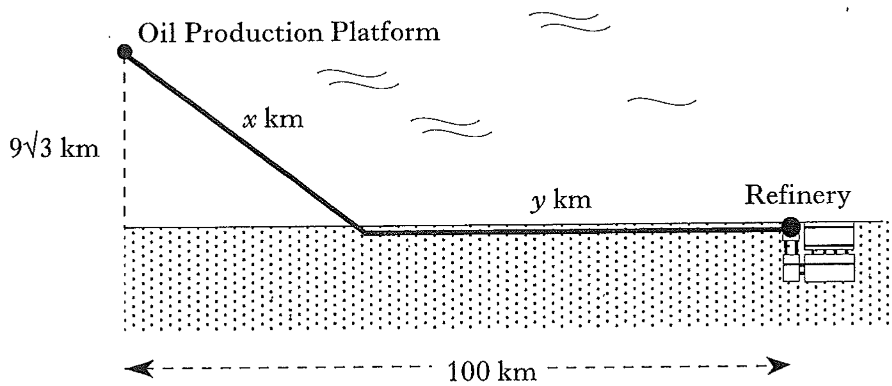
figure 2

(*a*) Find the equation of the line shown in figure 2 in terms of $\log_e I$ and $\log_e t$. **(3)**

(*b*) Hence or otherwise show that I and t satisfy a relationship of the form $I = kt^r$ stating the values of k and r. **(4)**

Marks

11. An oil production platform, $9\sqrt{3}$ km offshore, is to be connected by a pipeline to a refinery on shore, 100 km down the coast from the platform as shown in the diagram.



The length of underwater pipeline is x km and the length of pipeline on land is y km. It costs £2 million to lay each kilometre of pipeline underwater and £1 million to lay each kilometre of pipeline on land.

(*a*) Show that the total cost of this pipeline is £$C(x)$ million where

$$C(x) = 2x + 100 - \left(x^2 - 243\right)^{\frac{1}{2}}.$$ **(3)**

(*b*) Show that $x = 18$ gives a minimum cost for this pipeline.
Find this minimum cost and the corresponding total length of the pipeline. **(7)**

[END OF QUESTION PAPER]