Higher

Mathematics

HSNe20S01

Exam Solutions – 2001

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Paper 1

Question 1

Method 1. Required equation is of the form

$$2x + 3y = c.$$

Since $(2,-1)$ lies on the line

$$2(2) + 3(-1) = c$$

$$4 - 3 = c$$

$$\therefore c = 1$$

$\therefore$ Equation is $2x + 3y = 1$

$(\text{or } 2x + 3y - 1 = 0).$

Method 2. From $2x + 3y = 5$

$$\Leftrightarrow \quad 3y = -2x + 5$$

$$y = -\tfrac{2}{3}x + \tfrac{5}{3}$$

$$\therefore \quad m = -\tfrac{2}{3}$$

so $m_{\text{PARALLEL}} = -\tfrac{2}{3}$

$\therefore$ Equation of required line is

$$y - b = m(x - a)$$

$$y - (-1) = -\tfrac{2}{3}(x - 2)$$

$\times 3 \quad 3(y+1) = -2(x-2)$

$$3y + 3 = -2x + 4$$

$$\therefore \quad 2x + 3y - 1 = 0.$$

Question 2

Given $x^2 - 5x + (k+6) = 0$.

For equal roots $b^2 - 4ac = 0$ where $a = 1$, $b = -5$, $c = k+6$

$$\therefore (-5)^2 - 4 \times 1 \times (k+6) = 0$$
$$25 - 4(k+6) = 0$$
$$25 - 4k - 24 = 0$$
$$1 - 4k = 0$$
$$4k = 1$$
$$k = \frac{1}{4}.$$

Question 3

(a) $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} -8 \\ -10 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

$\overrightarrow{BC} = \underset{\sim}{c} - \underset{\sim}{b} = \begin{pmatrix} 6 \\ 11 \\ 5 \end{pmatrix} - \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

$$\therefore \overrightarrow{AB} = \frac{3}{4}\overrightarrow{BC}$$

Since $\overrightarrow{AB}$ and $\overrightarrow{BC}$ share a common direction and a common point B was used then A, B and C are collinear.

(b) $\overrightarrow{DB} = \underset{\sim}{b} - \underset{\sim}{d} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -4 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix}$

$\therefore \overrightarrow{AB}.\overrightarrow{DB} = 6(-3) + 9(3) + 3(-3) = -18 + 27 - 9 = 0$

Since $\overrightarrow{AB}.\overrightarrow{DB} = 0$

then AB is perpendicular to DB.

Question 4

$$f(x) = x^2 + 2x - 8$$
$$= (x^2 + 2x) - 8$$
$$= (x^2 + 2x + 1) - 8 - 1$$
$$= (x+1)^2 - 9$$

$x^2 + 2ax$.

Half the coefficient of x term and square the answer to give

$(x^2 + 2ax + a^2) - a^2$

$= (x+a)^2 - a^2$

Question 5

(a)

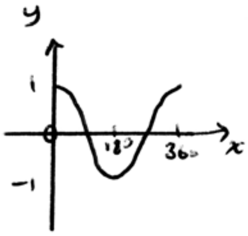
$\sin 2x° - \cos x° = 0$

Replace $\sin 2x$ by $2\sin x \cos x$

$2\sin x° \cos x° - \cos x° = 0$

$\cos x° (2\sin x° - 1) = 0$

$\therefore \cos x = 0$ or $\sin x = \frac{1}{2}$



From the $y = \cos x°$ graph

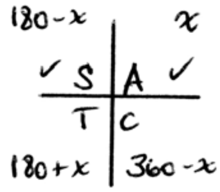
$x = 90$ or 270

$x = \sin^{-1}\left(\frac{1}{2}\right)$

$= 30$

or $180 - 30$

$= 30$ or 150



Since $0 \leq x \leq 180$

Solutions are $\{30, 90, 150\}$

(b)

At P $x = 150$

so $y = \cos 150°$

$= -\cos 30° = -\frac{\sqrt{3}}{2}$

Substitute $x = 150$ into either $y = \cos x°$ or $y = \sin 2x°$

$\therefore$ At $P\left(150, -\frac{\sqrt{3}}{2}\right)$.

Question 6

Given $P = 12x^3 - x^4$

$$\Rightarrow \frac{dP}{dx} = 36x^2 - 4x^3 = 4x^2(9-x)$$

Maximum profit occurs at stationary point (or end-point).

At stationary point(s)...

$$\frac{dP}{dx} = 0$$

$$\therefore \quad 4x^2(9-x) = 0$$

$$x = 0 \quad \text{or} \quad x = 9.$$

When $x = 9$

x	9^-	9	9^+
$\frac{dP}{dx}$	$+$	0	$-$
Shape	/	—	\

Maximum value occurs when $x = 9$

$\therefore$ Maximum profit occurs when $x = 9$.

Question 7

(a) (i) $f(h(x)) = f\left(x + \frac{\pi}{4}\right) = \sin\left(x + \frac{\pi}{4}\right)$

(ii) $g(h(x)) = g\left(x + \frac{\pi}{4}\right) = \cos\left(x + \frac{\pi}{4}\right)$

(b) (i)

$$f(h(x)) = \sin\left(x + \frac{\pi}{4}\right)$$

$$= \sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}$$

$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$= \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x.$$ as required.

(ii)

$$g(h(x)) = \cos\left(x + \frac{\pi}{4}\right)$$

$$= \cos x \cos\frac{\pi}{4} - \sin x \sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x$$

$$\therefore\ f(h(x)) - g(h(x)) = 1$$

$$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x - \left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right) = 1$$

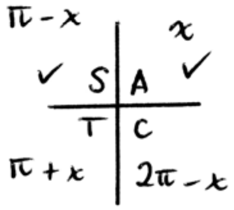
$$\frac{1}{\sqrt{2}}\sin x + \cancel{\frac{1}{\sqrt{2}}\cos x} - \cancel{\frac{1}{\sqrt{2}}\cos x} + \frac{1}{\sqrt{2}}\sin x = 1$$

$$\frac{2}{\sqrt{2}}\sin x = 1$$

$\frac{2}{\sqrt{2}} = \sqrt{2}$

$$\sqrt{2}\sin x = 1$$

$$\sin x = \frac{1}{\sqrt{2}}$$



$$x = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi}{4} \text{ or } \pi - \frac{\pi}{4}$$

$$= \frac{\pi}{4} \text{ or } \frac{3\pi}{4}.$$

Question 8

$$4\log_x 6 - 2\log_x 4 = 1$$

$$\therefore \log_x 6^4 - \log_x 4^2 = 1$$

$$\Leftrightarrow \log_x \frac{6^4}{4^2} = 1$$

$$\Leftrightarrow x^1 = \frac{6^4}{4^2}$$

$$\therefore x = 81.$$

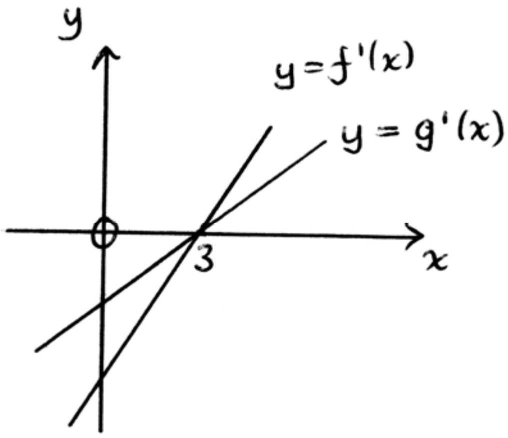
Using the rules of logs...

$$n\log_a p = \log_a p^n$$

$$\log_a p - \log_a q = \log_a \frac{p}{q}$$

$$\frac{6^4}{4^2} = \frac{6^2}{4} \times \frac{6^2}{4} = 9 \times 9 = 81$$

Question 9



Stationary point at $x = 3$

$\therefore$ graph of derivative cuts x-axis at $x = 3$

Question 10

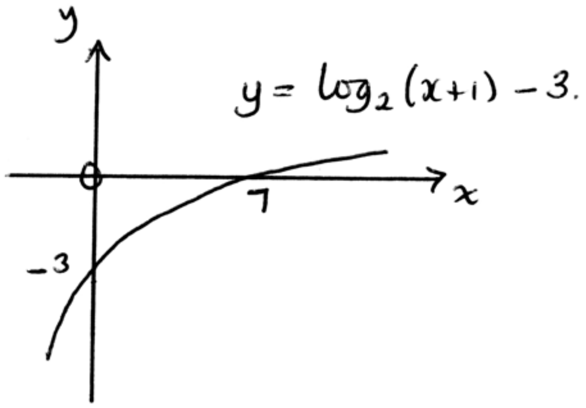
(a) Given $y = \log_2 x$.

When $\left.\begin{array}{l} x = a \\ y = 0 \end{array}\right\}$ $\log_2 a = 0 \Leftrightarrow a = 2^0 = 1$.

When $\left.\begin{array}{l} y = b \\ x = 8 \end{array}\right\}$ $\log_2 8 = b \Leftrightarrow 2^b = 8 = 2^3$

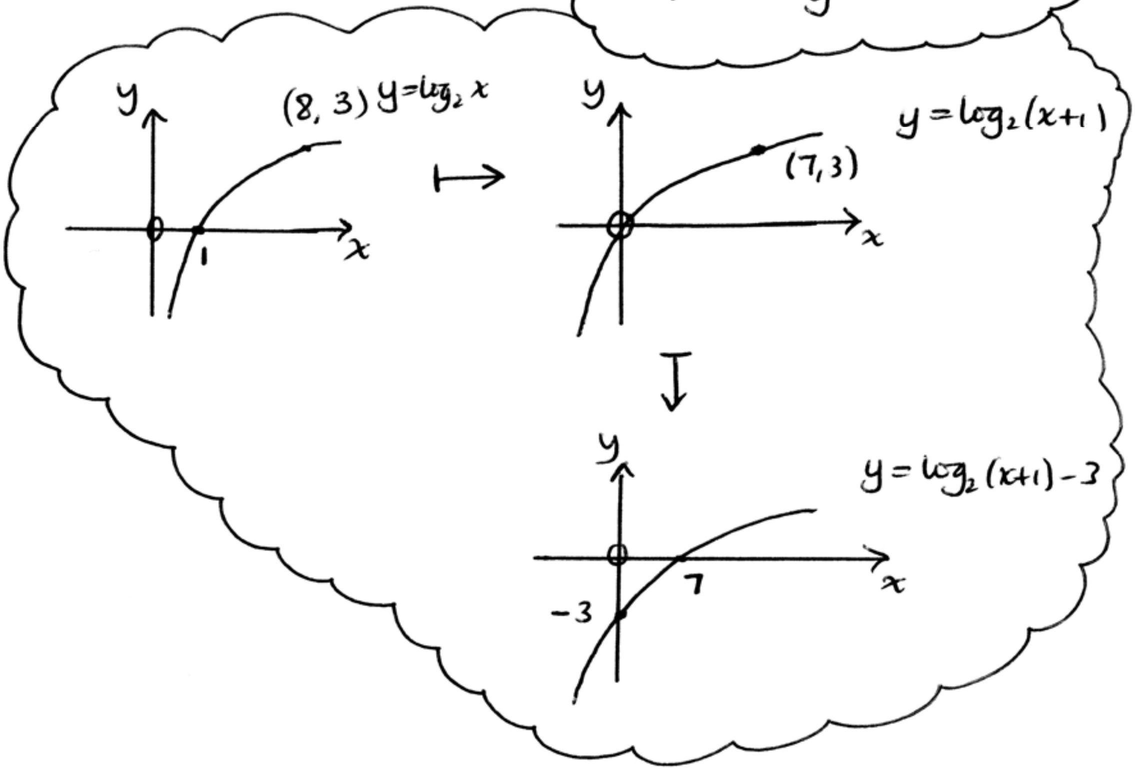
$\therefore b = 3$

(b)



This is the graph of $y = \log_2 x$ translated…

- 1 unit to the left parallel to the x-axis
- 3 units down parallel to the y-axis



Question 11

(a) (i) Circle P: $x^2+y^2-8x-10y+9=0$

Centre $(4, 5)$

Radius $= \sqrt{4^2+5^2-9}$

$= \sqrt{16+25-9}$

$= \sqrt{32} = \sqrt{16 \times 2}$

$= 4\sqrt{2}$ units.

(ii) Distance between centres of circles

$= \sqrt{(4-(-2))^2 + (5-(-1))^2}$

$= \sqrt{6^2+6^2}$

$= \sqrt{72}$

$= \sqrt{36 \times 2} = 6\sqrt{2}$ units.

Sum of radii $= 4\sqrt{2} + 2\sqrt{2} = 6\sqrt{2}$ units.

Since sum of radii = distance between centres of circles

$\therefore$ Circles P and Q touch.

(b) $m_{\text{RADIUS}} = \dfrac{-1-1}{-2-(-4)} = \dfrac{-2}{2} = -1$

$\therefore \quad m_{\text{TANGENT}} = 1$

Since radius and tangent are perpendicular.

$\therefore$ Equation of tangent is

$$y-b = m(x-a)$$
$$y-1 = 1\,(x-(-4))$$
$$y-1 = x+4$$
$$x-y+5=0$$

(c) Substituting eq^{n} of tangent

$$y = x+5$$

Into equation of circle P...

$$x^2+y^2-8x-10y+9=0$$

then...

$$x^2+(x+5)^2-8x-10(x+5)+9=0$$
$$x^2+x^2+10x+25-8x-10x-50+9=0$$
$$\therefore \quad 2x^2-8x-16=0$$
$$\div 2 \quad x^2-4x-8=0$$

This does not factorise since b^2-4ac is not a perfect square.

Method 1. Using the quadratic formula...

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{where} \quad \begin{array}{l} a = 1 \\ b = -4 \\ c = -8 \end{array}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 1 \times (-8)}}{2 \times 1}$$

$$= \frac{4 \pm \sqrt{16+32}}{2} = \frac{4 \pm \sqrt{48}}{2}$$

$\sqrt{48} = \sqrt{16 \times 3}$
$= 4\sqrt{3}$

$$= \frac{4 \pm 4\sqrt{3}}{2}$$

$$= \frac{\cancel{2}(2 \pm 2\sqrt{3})}{\cancel{2}} = 2 \pm 2\sqrt{3}$$

Method 2 Completing the square...

$$x^2 - 4x - 8 = 0$$

$$\therefore \quad x^2 - 4x = 8$$

$$x^2 - 4x + 4 = 8 + 4$$

$$(x-2)^2 = 12$$

$$x - 2 = \pm\sqrt{12}$$

$$x - 2 = \pm 2\sqrt{3}$$

$$x = 2 \pm 2\sqrt{3}.$$

Half -4 and square the answer
$x^2 - 4x \ldots$
$x^2 - 4x + (-2)^2$
$= (x-2)^2$

$\sqrt{12}$
$= \sqrt{4 \times 3}$
$= 2\sqrt{3}.$

Paper 2

Question 1

(a) Since $x+2$ is a factor then $x=-2$ is a root.

$$\begin{array}{r|rrrr} -2 & 2 & 1 & k & 2 \\ & & -4 & 6 & -2k-12 \\ \hline & 2 & -3 & k+6 & -2k-10 \end{array}$$

remainder

Remainder is 0

$\therefore\ -2k-10=0$

$-2k=10$

$k=-5$

(b) When $k=-5$ then $2x^3+x^2-5x+2=0$

$(x+2)(2x^2-3x+1)=0$

From (a) above

$(x+2)(x-1)(2x-1)=0$

$\therefore\ x+2=0$ or $x-1=0$ or $2x-1=0$

$x=-2$ or $x=1$ or $x=\frac{1}{2}$

So solutions are $\left\{-2, \frac{1}{2}, 1\right\}$

Question 2

$y = x - \dfrac{16}{\sqrt{x}} = x - 16x^{-\frac{1}{2}}$

$\therefore\ m_{tangent} = \dfrac{dy}{dx} = 1 + 8x^{-3/2} = 1 + \dfrac{8}{x^{3/2}}$.

When $x=4$ $\quad y = 4 - \dfrac{16}{\sqrt{4}} = 4 - 8 = -4$

and $m_{tangent} = 1 + \dfrac{8}{4^{3/2}} = 1 + \dfrac{8}{8} = 1+1 = 2$

So equation of tangent is $y-b=m(x-a)$

$y-(-4)=2(x-4)$

$y+4=2x-8$

$2x-y-12=0$

$4^{3/2}=(\sqrt{4})^3$
$=2^3=8$

Question 3

(a) $u_{n+1} = 1.015u_n - 300 \qquad u_0 = 2500$

(b)

before interest is added

Date	Amount Outstanding
April 1st	£2237.50
May 1st	£1971.06
June 1st	£1700.63
July 1st	£1426.14
August 1st	£1147.53
September 1st	£864.74
October 1st	£577.71
November 1st	£286.38

On your calculator...

2500 $\boxed{=}$

1.015 × $\boxed{\text{Ans}}$ − 300 $\boxed{=}$

Each time you press $\boxed{=}$ the next answer will be displayed.

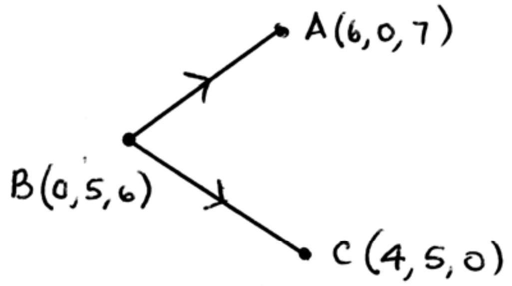
Payment due on 1st December

£286.38 + 1.5% interest

ie £290.68

Question 4

Let $\angle ABC$ be θ...



$$\overrightarrow{BA} = \underset{\sim}{a} - \underset{\sim}{b} = \begin{pmatrix} 6 \\ 0 \\ 7 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ 1 \end{pmatrix}$$

$$\overrightarrow{BC} = \underset{\sim}{c} - \underset{\sim}{b} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -6 \end{pmatrix}$$

$$\begin{aligned} \overrightarrow{BA} \cdot \overrightarrow{BC} &= 6 \times 4 + (-5 \times 0) + 1 \times (-6) \\ &= 24 + 0 - 6 \\ &= 18 \end{aligned}$$

$$|\overrightarrow{BA}| = \sqrt{6^2 + (-5)^2 + 1^2} = \sqrt{36 + 25 + 1} = \sqrt{62}$$

$$|\overrightarrow{BC}| = \sqrt{4^2 + 0^2 + (-6)^2} = \sqrt{16 + 0 + 36} = \sqrt{52}$$

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BC}| \cos\theta$$

$$\therefore \quad \cos\theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \frac{18}{\sqrt{62}\sqrt{52}}$$

So $\theta = \cos^{-1}\left(\frac{18}{\sqrt{3224}}\right)$

$= 71 \cdot 5^\circ$ (to 1dp)

$\underline{\underline{\text{or}}}$ $1 \cdot 248^c$ (to 3dp)

Question 5

$$8\cos x^\circ - 6\sin x^\circ = k\cos(x+a)^\circ$$
$$= k\cos x\cos a - k\sin x\sin a$$

Equating coefficients $\left.\begin{array}{l} k\cos a = 8 \\ k\sin a = 6 \end{array}\right\} k = \sqrt{8^2+6^2} = \sqrt{100} = 10$.

a is in 1st quadrant $\dfrac{k\sin a}{k\cos a} = \tan a = \dfrac{6}{8} = \dfrac{3}{4}$

$$\therefore a = \tan^{-1}\left(\tfrac{3}{4}\right) = 36{\cdot}9^\circ$$
(to 1dp)

$$\therefore 8\cos x^\circ - 6\sin x^\circ = 10\cos(x+36{\cdot}9)^\circ$$

Question 6

$$\int \frac{(x^2-2)(x^2+2)}{x^2}\,dx$$

$(x^2-2)(x^2+2) \leftarrow$ difference of two squares.

$$= (x^2)^2 - 2^2$$
$$= x^4 - 4$$
$$\therefore \frac{x^4-4}{x^2} = \frac{x^4}{x^2} - \frac{4}{x^2}$$
$$= x^2 - 4x^{-2}$$

$$= \int x^2 - 4x^{-2}\,dx$$
$$= \tfrac{1}{3}x^3 + 4x^{-1} + c$$
$$= \tfrac{1}{3}x^3 + \frac{4}{x} + c.$$

Question 7

(a) Mid-point of AB is $(7, 2)$.

$\therefore$ Equation of perpendicular bisector is

$$x = 7.$$

Since AB is parallel to x-axis then perpendicular is parallel to the y-axis

(b) Mid-point of AC is $(5,4)$

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{6-2}{8-2} = \frac{4}{6} = \frac{2}{3}$$

$$\therefore\ m_{PERP} = -\frac{3}{2}$$

So equation of perpendicular bisector is

$$y - b = m(x-a)$$

$$y - 4 = -\frac{3}{2}(x-5)$$

$\times 2$ $\quad 2y - 8 = -3x + 15$

$$\therefore\ 3x + 2y - 23 = 0$$

(c)

$x = 7$ ——— ①

$3x + 2y - 23 = 0$ ——— ②

Substitute ① into ②

$$21 + 2y - 23 = 0$$

$$2y = 2$$

$$y = 1$$

$\therefore$ Point of intersection is $(7,1)$

(d) Centre of circle is $(7,1)$.

Radius $= \sqrt{(8-7)^2 + (6-1)^2}$ Using C

$= \sqrt{1^2 + 5^2} = \sqrt{26}$

Radius of circle is distance from centre to A, B or C

$\therefore$ Equation of circle is

$$(x-a)^2 + (y-b)^2 = r^2$$

$$\therefore\ (x-7)^2 + (y-1)^2 = 26$$

Question 8

Since areas are symmetrical.

Total shaded area $= 2\int_{-2}^{1} y_{\text{CURVE}} - y_{\text{LINE}} \; dx$

Using $\int_a^b$ upper - lower ...

$$= 2\int_{-2}^{1} (x^3 - 3x^2 - x + 3) - (5x - 5)\, dx$$

$$(x+1)(x-1)(x-3)$$
$$= (x^2-1)(x-3)$$
$$= x^3 - x - 3x^2 + 3$$
$$= x^3 - 3x^2 - x + 3$$

$$= 2\int_{-2}^{1} x^3 - 3x^2 - x + 3 - 5x + 5 \; dx$$

$$= 2\int_{-2}^{1} x^3 - 3x^2 - 6x + 8 \; dx$$

$$= 2\left[\tfrac{1}{4}x^4 - x^3 - 3x^2 + 8x\right]_{-2}^{1}$$

$$= 2\left[\left(\tfrac{1}{4} - 1 - 3 + 8\right) - \left(\tfrac{1}{4}(-2)^4 - (-2)^3 - 3(-2)^2 + 8(-2)\right)\right]$$

$$= 2\left[4\tfrac{1}{4} - (4 + 8 - 12 - 16)\right]$$

$$= 2\left(4\tfrac{1}{4} - (-16)\right)$$

$$= 2 \times 20\tfrac{1}{4}$$

$$= 40\tfrac{1}{2} \text{ square units}$$

Question 9

Given $A = A_0 e^{kt}$

When $t = 1.5$ $A = 2A_0$ ie double the area covered

$$\therefore \quad 2\cancel{A_0} = \cancel{A_0} e^{1.5k}$$

$$\text{so} \quad e^{1.5k} = 2$$

taking $\log_e$ of both sides...

$$\log_e e^{1.5k} = \log_e 2$$

$$\therefore \quad 1.5k \log_e e = \log_e 2$$

$$1.5k = \log_e 2$$

$$k = \frac{\log_e 2}{1.5}$$

$$= 0.462 \text{ to 3dp.}$$

If you can you can go straight to these lines in the exam!

Remember...
$\log_a p^n = n \log_a p$
$\log_a a = 1$

$\log_e 2$ on your calculator is $\ln 2$

Question 10

Given $\dfrac{dy}{dx} = 3\sin 2x$

$$\therefore \quad y = \int 3\sin 2x \, dx$$

$$= -\frac{3}{2}\cos 2x + c.$$

Remember...
$\frac{5}{12}\pi \equiv \frac{5\pi}{12}$

When $\left.\begin{array}{l} x = \frac{5}{12}\pi \\ y = \sqrt{3} \end{array}\right\}$ $\sqrt{3} = -\frac{3}{2}\cos\left(2 \cdot \frac{5\pi}{12}\right) + c$

$$= -\frac{3}{2}\cos\frac{5\pi}{6} + c$$

$$= -\frac{3}{2}\left(-\frac{\sqrt{3}}{2}\right) + c = \frac{3\sqrt{3}}{4} + c$$

$\cos\frac{5\pi}{6} \equiv \cos 150^\circ$
$= -\cos 30^\circ$
$= -\frac{\sqrt{3}}{2}$

So $c = \sqrt{3} - \frac{3}{4}\sqrt{3} = \frac{1}{4}\sqrt{3}$ or $\frac{\sqrt{3}}{4}$ $\quad \therefore \; y = -\frac{3}{2}\cos 2x + \frac{\sqrt{3}}{4}.$

Question 11

(a) Roots are $x = -1$ and $x = p$

$\therefore$ Equation of the parabola is of the form

$$y = k(x+1)(x-p).$$

Since $(0, p)$ lies on the parabola...

$$p = k \times 1 \times (-p)$$

$$\therefore \quad -kp = p$$

$$k = -1.$$

$\therefore$ Equation is

$$\begin{aligned} y &= -(x+1)(x-p) \\ &= -(x^2 - px + x - p) \\ &= -x^2 + px - x + p \\ &= p + (p-1)x - x^2 \text{ as required.} \end{aligned}$$

(b) At point of intersection...

$$y_{\text{PARABOLA}} = y_{\text{TANGENT}}$$

$$p + (p-1)x - x^2 = x + p$$

$$\therefore \quad x^2 + x - (p-1)x + \cancel{p} - \cancel{p} = 0$$

$$x^2 + (1 - p + 1)x = 0$$

$$x^2 + (2-p)x = 0$$

however since the line is a tangent to the circle $b^2 - 4ac = 0$

Using $b^2 - 4ac = 0$

$$(2-p)^2 - 4 \times 1 \times 0 = 0$$

$$(2-p)^2 = 0$$

$$\text{so } p = 2$$