Higher

# Mathematics

HSNe20S00

Exam Solutions – 2000 (Amended)

## Contents

# Paper 1

## Question 1

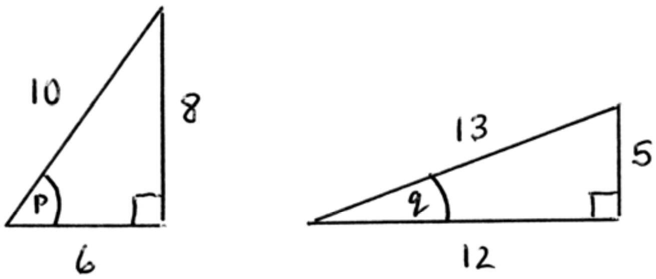
$$f(x) = x^4 - \frac{4}{x} = x^4 - 4x^{-1}$$

$$\therefore f'(x) = 4x^3 + 4x^{-2} = 4x^3 + \frac{4}{x^2}$$

$$\text{So } f'(-2) = 4(-2)^3 + \frac{4}{(-2)^2}$$

$$= 4 \times (-8) + \frac{4}{4} = -32 + 1 = -31$$

## Question 2




Both triangles contain Pythagorean Triples (6, 8, 10) and (5, 12, 13)

$$\sin p = \frac{8}{10} = \frac{4}{5} \qquad \cos p = \frac{6}{10} = \frac{3}{5}$$

$$\sin(p+q) = \sin p \cos q + \cos p \sin q$$

$$= \frac{4}{5} \times \frac{12}{13} + \frac{3}{5} \times \frac{5}{13}$$

$$= \frac{48}{65} + \frac{15}{65}$$

$$= \frac{63}{65}.$$

## Question 3

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{2-(-1)}{3\sqrt{3}-0}$$

$$= \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

however $\tan a^\circ = m_{AB}.$

$\therefore \quad \tan a^\circ = \dfrac{1}{\sqrt{3}}$

so $a^\circ = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = 30^\circ$

## Question 4

(a) Given $f(x) = x^3 - 6x^2 + 9x.$

$\Rightarrow f'(x) = 3x^2 - 12x + 9$

Stationary points occur when $f'(x) = 0$

$\therefore \quad 3x^2 - 12x + 9 = 0$

$3(x^2 - 4x + 3) = 0$

$(\div 3) \quad x^2 - 4x + 3 = 0$

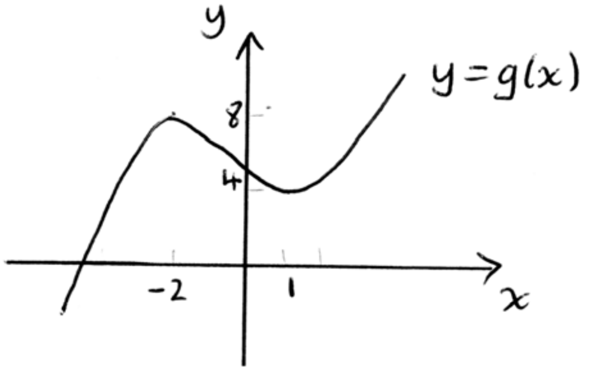
$(x-3)(x-1) = 0$

$\therefore \quad x = 3$ or $x = 1.$

At A $x = 1$ and $y = f(1) = 1^3 - 6(1^2) + 9(1) = 1 - 6 + 9 = 4$

$\therefore$ Coordinates of A are $(1, 4).$

(b)




$g(x) = f(x+2) + 4$

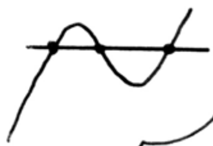
Graph of $g(x)$ is the graph of $f(x)$ translated 2 units to the left parallel to the $x$-axis, and 4 units up parallel to the $y$-axis.

$A(1, 4) \mapsto A'(-2, 8)$

$B(3, 0) \mapsto B'(1, 4)$.

(c) $4 < k < 8$

$y = k$ is a horizontal line which cuts the above graph at 3 different places



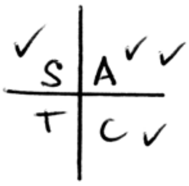
## Question 5

**(a)** $\cos x - \sin x = k\cos(x+a)$

$$= k\cos x\cos a + k\sin x\sin a$$

Equating coefficients...

$$\left.\begin{aligned} k\cos a &= 1 \\ k\sin a &= 1 \end{aligned}\right\} \quad k = \sqrt{1^2+1^2} = \sqrt{2}$$




$a$ is in 1st quadrant.

$$\frac{\cancel{k}\sin a}{\cancel{k}\cos a} = \tan a = \frac{1}{1} = 1$$

$$\therefore \quad a = \tan^{-1} 1 = \frac{\pi}{4}.$$

$$\therefore \quad \cos x - \sin x = \sqrt{2}\cos\left(x+\frac{\pi}{4}\right)$$

**(b)** Since $\cos x - \sin x = \sqrt{2}\cos\left(x+\frac{\pi}{4}\right)$

the maximum value of $\cos x - \sin x$ is $\sqrt{2}$

When $\cos\left(x+\frac{\pi}{4}\right) = 1$

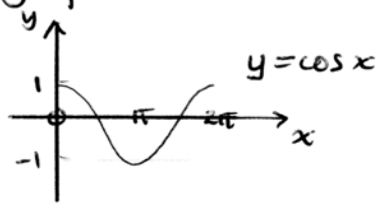
$$\therefore \quad x + \frac{\pi}{4} = 0 \text{ or } 2\pi$$

$$x = -\frac{\pi}{4} \text{ or } \frac{7\pi}{4}$$

Since $0 \le x \le 2\pi$

then maximum value of $\sqrt{2}$ occurs when $x = \frac{7\pi}{4}$.

You should know this from the cosine graph...



## Question 6

$$\log_5 2 + \log_5 50 - \log_5 4$$

$$= \log_5 (2 \times 50) - \log_5 4$$

$$= \log_5 100 - \log_5 4$$

$$= \log_5 \tfrac{100}{4}$$

$$= \log_5 25 \quad \underline{\underline{\text{OR}}} \quad \log_5 5^2$$

$$= 2 \qquad\qquad = 2\log_5 5$$

$$= 2$$

Using the laws of logs...

$\log_a p + \log_a q = \log_a (pq)$

$\log_a p - \log_a q = \log_a \left(\frac{p}{q}\right)$

## Question 7

$$y = f(x) = \int f'(x)\,dx = \int \sin 3x \,dx$$

$$= -\tfrac{1}{3}\cos 3x + c.$$

$\cos\frac{\pi}{3} \equiv \cos 60^\circ$
$= \frac{1}{2}$

When $\left.\begin{matrix} x = \frac{\pi}{9} \\ y = 1 \end{matrix}\right\}$ $1 = -\frac{1}{3}\cos 3\left(\frac{\pi}{9}\right) + c = -\frac{1}{3}\cos\frac{\pi}{3} + c$

$$\therefore \; -\tfrac{1}{3} \times \tfrac{1}{2} + c = 1 \Rightarrow -\tfrac{1}{6} + c = 1 \Rightarrow c = \tfrac{7}{6}$$

So $y = -\frac{1}{3}\cos 3x + \frac{7}{6}$

## Question 8

$x^2 + y^2 + 4x - 2y + k = 0$

will represent a circle if

$g^2 + f^2 - c > 0$ where $g = 2$, $f = -1$, $c = k$.

$\therefore \quad 2^2 + (-1)^2 - k > 0$

$5 - k > 0$

$k < 5.$

This equation will represent a circle when

$\{ k : k < 5, k \in \mathbb{R} \}$

## Question 9

From the diagram

$\overrightarrow{VK} = \overrightarrow{VA} + \overrightarrow{AB} + \overrightarrow{BK}$ where $\overrightarrow{BK} = \frac{1}{4}\overrightarrow{BC}$ since $\overrightarrow{BC} = \overrightarrow{AD}$

$= \frac{1}{4}\overrightarrow{AD}$

$= \frac{1}{4}\begin{pmatrix} 8 \\ -4 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -7 \\ -13 \\ -11 \end{pmatrix} + \begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -7 + 6 + 2 \\ -13 + 6 - 1 \\ -11 - 6 + 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -8 \\ -16 \end{pmatrix}$

## Question 10

(a) (i) From $y = x^2 - 4x + 3$

$$m_{tangent} = \frac{dy}{dx} = 2x - 4$$

At $x = a$ $m_{tangent} = 2a - 4$

(ii) From $y = x^3 - 2x^2 - 4x + 3$

$$m_{tangent} = 3x^2 - 4x - 4$$

At $x = a$ $m_{tangent} = 3a^2 - 4a - 4$

(iii) When these gradients are equal...

$$3a^2 - 4a - 4 = 2a - 4$$

$$\therefore \quad 3a^2 - 4a - 4 - 2a + 4 = 0$$

$$3a^2 - 6a = 0$$

$$3a(a - 2) = 0$$

$$\therefore \quad a = 0 \quad \text{or} \quad a = 2.$$

(iv) When $a = 0$ $m_{tangent} = -4$ This corresponds to point A on the graph. (Common tangent to the point of intersection of the curves)

When $a = 2$ $m_{tangent} = 0$ so a stationary point

This corresponds to the minimum turning points on both curves.

(b)

$$\text{Area enclosed} = \int_0^3 (x^2 - 4x + 3) - (x^3 - 2x^2 - 4x + 3)\, dx$$

Using
Area = $\int$ upper − lower

$$= \int_0^3 x^2 \cancel{- 4x} \cancel{+ 3} - x^3 + 2x^2 \cancel{+ 4x} \cancel{- 3}\, dx$$

$$= \int_0^3 -x^3 + 3x^2\, dx.$$

$$= \int_0^3 3x^2 - x^3\, dx$$

$$= \left[ x^3 - \frac{1}{4}x^4 \right]_0^3$$

$$= \left(3^3 - \frac{1}{4}.\, 3^4\right) - 0$$

since each term contains an $x$.

$$= 27 - \frac{81}{4}$$

$$= 27 - 20\tfrac{1}{4}$$

$$= 6\tfrac{3}{4} \text{ or } \frac{27}{4} \text{ square units}$$

## Question 11

From $u_{n+1} = au_n + 10$.

Method 1. Using $l = \frac{b}{1-a}$

$= \frac{10}{1-a}$.

OR Method 2 as $n \to \infty$

$u_{n+1} = u_n = l$

$\therefore \; l = al + 10$

$l - al = 10$

$(1-a)l = 10$

$\therefore \; l = \frac{10}{1-a}$.

Similarly, from $v_{n+1} = a^2 v_n + 16$.

Method 1. Using $l = \frac{b}{1-a}$

$= \frac{16}{1-a^2}$

OR Method 2

as $n \to \infty$ $v_{n+1} = v_n = l$

$\therefore \; l = a^2 l + 16$

$l - a^2 l = 16$

$(1-a^2)l = 16$

$\therefore \; l = \frac{16}{1-a^2}$

Since both limits are equal ...

$$\frac{10}{1-a} = \frac{16}{1-a^2}$$

Cross-multiplying $10(1-a^2) = 16(1-a)$

$10 - 10a^2 = 16 - 16a$

$\therefore \; 10a^2 - 16a + 6 = 0$

$\div 2$ $\quad 5a^2 - 8a + 3 = 0$

$(5a-3)(a-1) = 0$

$\therefore \; a = \frac{3}{5}$ or $a = 1$

For a limit to exist $-1 < a < 1$ so reject $a = 1$

$\therefore \; a = 3/5$ and limit $= \frac{10}{1-3/5} = \frac{10}{2/5} = 25$

$\frac{10}{2/5} = \cancel{10}^{5} \times \frac{5}{\cancel{2}_{1}}$

$= 5 \times 5$

# Paper 2

## Question 1

**(a)** Given $y = x^3 - 3x^2 + 2x$

$$\Rightarrow m_{tangent} = \frac{dy}{dx} = 3x^2 - 6x + 2$$

When $x = 1$ $\quad y = 1^3 - 3(1^2) + 2(1) = 1 - 3 + 2 = 0.$

$$m_{tangent} = 3(1^2) - 6(1) + 2 = 3 - 6 + 2 = -1$$

$\therefore$ Equation of tangent is

$$y - b = m(x - a)$$

$$y - 0 = -1(x - 1)$$

$$y = -x + 1$$

$$\therefore \quad x + y - 1 = 0.$$

**(b)** At points of intersection

$$y_{\text{CURVE}} = y_{\text{TANGENT}}$$

$$x^3 - 3x^2 + 2x = 2x - 4$$

$$\therefore \quad x^3 - 3x^2 + 2x - 2x + 4 = 0$$

$$x^3 - 3x^2 + 4 = 0$$

$$\therefore \quad (x-2)(x^2 - x - 2) = 0$$

$$(x-2)(x-2)(x+1) = 0$$

$$(x-2)^2(x+1) = 0$$

$$\therefore \quad x = 2 \quad \text{or} \quad x = -1$$

Tangent meets curve again at $x = -1$

so $y = 2(-1) - 4 = -2 - 4 = -6$

ie at the point $(-1, -6)$.

$x = 2$ is a root of this equation

Using Synthetic Division

$$\begin{array}{r|rrrr}
2 & 1 & -3 & 0 & 4 \\
 & & 2 & -2 & -4 \\
\hline
 & 1 & -1 & -2 & \underline{\underline{0}}
\end{array}$$

since $x = 2$ is a root then $x - 2$ is a factor.

## Question 2

Since $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular then $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$

$$\begin{aligned} \underset{\sim}{u} \cdot \underset{\sim}{v} &= t \times 2 + (-2) \times 10 + 3 \times t \\ &= 2t - 20 + 3t \\ &= 5t - 20 \end{aligned}$$

$$\begin{aligned} \therefore \quad 5t - 20 &= 0 \\ 5t &= 20 \\ t &= 4 \end{aligned}$$

## Question 3

(a) Midpoint of PQ is $\left(\frac{-3+1}{2}, \frac{1+9}{2}\right) = (-1, 5)$

$$m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{9-1}{1-(-3)} = \frac{8}{4} = 2$$

$\therefore \; m_{AB} = -\frac{1}{2}$ (since PQ and AB are perpendicular)

So equation of AB is

$$\begin{aligned} y - b &= m(x - a) \\ y - 5 &= -\tfrac{1}{2}(x + 1) \\ \times 2 \qquad 2y - 10 &= -x - 1 \\ \therefore \quad x + 2y - 9 &= 0 \end{aligned}$$

(b) Since CQ is parallel to the y-axis $x_c = 1$
and as C lies on the line AB.

$$1 + 2y - 9 = 0$$
$$2y = 8$$
$$y = 4$$

$\therefore$ Coordinates of C are $(1, 4)$ ← Centre of circle

Radius of circle = distance from C to Q
= 5 units

$\therefore$ Equation of circle is

$$(x-1)^2 + (y-4)^2 = 5^2$$
ie $$(x-1)^2 + (y-4)^2 = 25$$

(c) (i) Tangent at Q is perpendicular to CQ
$\Rightarrow$ Tangent at Q is parallel to the x-axis

$\therefore$ Equation of tangent is
$$y = 9$$

(ii) At T $y = 9$

and since this point lies on AB

$$x + 2(9) - 9 = 0$$
$$x + 18 - 9 = 0$$
$$x = -9$$

$\therefore$ Coordinates of T are $(-9, 9)$

## Question 4

(a)
$$p(x) = f(g(x))$$
$$= f\left(\frac{3}{x}\right)$$
$$= 3 - \frac{3}{x}$$

(b)
$$p(q(x)) = p\left(\frac{3}{3-x}\right)$$
$$= 3 - \frac{3}{\frac{3}{3-x}}$$
$$= 3 - 3 \div \frac{3}{3-x}$$
$$= 3 - \cancel{3} \times \frac{3-x}{\cancel{3}}$$
$$= 3 - (3-x)$$
$$= 3 - 3 + x$$
$$= x.$$

## Question 5

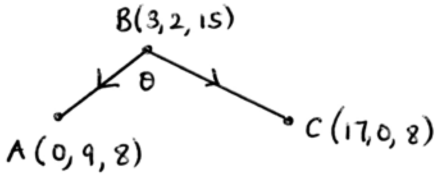
$$f(x) = (5x-4)^{\frac{1}{2}}$$
$$\Rightarrow f'(x) = \frac{1}{2}(5x-4)^{-1/2} \times 5$$
$$= \frac{5}{2}(5x-4)^{-1/2} = \frac{5}{2(5x-4)^{1/2}}$$

$$\therefore f'(4) = \frac{5}{2(20-4)^{1/2}} = \frac{5}{2 \times \sqrt{16}} = \frac{5}{2 \times 4} = \frac{5}{8}$$

## Question 6

**(a)** $B(3, 2, 15)$

**(b)**




Let $\angle ABC$ be $\theta$

$$\overrightarrow{BA}\cdot\overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BC}|\cos\theta$$

$$\therefore \cos\theta = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}$$

$$\overrightarrow{BA} = \underset{\sim}{a} - \underset{\sim}{b} = \begin{pmatrix}0\\9\\8\end{pmatrix} - \begin{pmatrix}3\\2\\15\end{pmatrix} = \begin{pmatrix}-3\\7\\-7\end{pmatrix}$$

$$\overrightarrow{BC} = \underset{\sim}{c} - \underset{\sim}{b} = \begin{pmatrix}17\\0\\8\end{pmatrix} - \begin{pmatrix}3\\2\\15\end{pmatrix} = \begin{pmatrix}14\\-2\\-7\end{pmatrix}$$

$$\begin{aligned}\overrightarrow{BA}\cdot\overrightarrow{BC} &= (-3)\times 14 + 7\times(-2) + (-7)\times(-7)\\ &= -42 - 14 + 49\\ &= -7\end{aligned}$$

$$|\overrightarrow{BA}| = \sqrt{(-3)^2 + 7^2 + (-7)^2} = \sqrt{9 + 49 + 49} = \sqrt{107}$$

$$|\overrightarrow{BC}| = \sqrt{14^2 + (-2)^2 + (-7)^2} = \sqrt{196 + 4 + 49} = \sqrt{249}$$

$$= \frac{-7}{\sqrt{107}\sqrt{249}}$$

$$\therefore \theta = 180^\circ - \cos^{-1}\left(\frac{7}{\sqrt{107\times 249}}\right)$$

$$= \underline{\underline{92\cdot 5^\circ}} \text{ (to 1 dp)}$$

## Question 7

$$\int \frac{1}{(7-3x)^2}\,dx = \int (7-3x)^{-2}\,dx$$

$$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$$

$$= \frac{1}{-3(-2+1)}(7-3x)^{-2+1} + c$$

$$= \frac{1}{-3\times(-1)}(7-3x)^{-1} + c = \frac{1}{3}(7-3x)^{-1} + c$$

$$= \frac{1}{3(7-3x)} + c$$

## Question 8

$$A(x) = \frac{3\sqrt{3}}{2}\left(x^2 + \frac{16}{x}\right) = \frac{3\sqrt{3}}{2}\left(x^2 + 16x^{-1}\right)$$

$$\therefore \ A'(x) = \frac{3\sqrt{3}}{2}\left(2x - 16x^{-2}\right)$$

At minimum value $A'(x) = 0$

$$\therefore \ \frac{3\sqrt{3}}{2}\left(2x - 16x^{-2}\right) = 0$$

$$\div \frac{3\sqrt{3}}{2} \qquad 2x - \frac{16}{x^2} = 0$$

$$2x = \frac{16}{x^2}$$

$$2x^3 = 16$$

$$x^3 = 8$$

$$\therefore \ x = \sqrt[3]{8} = 2$$

Nature

Method 1. Table of Values

| $x$ | $2^-$ | $2$ | $2^+$ |
|---|---|---|---|
| $A'(x) = \frac{3\sqrt{3}}{2}\left(2x - \frac{16}{x^2}\right)$ | $-$ | $0$ | $+$ |
| Shape | \ | — | / |

Minimum value when $x = 2$.

Method 2 Second derivative

$$A''(x) = \frac{3\sqrt{3}}{2}\left(2 + 32x^{-3}\right) = \frac{3\sqrt{3}}{2}\left(2 + \frac{32}{x^3}\right)$$

$$\therefore \ A''(2) = \frac{3\sqrt{3}}{2}\left(2 + \frac{32}{8}\right) = \frac{3\sqrt{3}}{2} \times 6 = 9\sqrt{3} > 0$$

$\therefore$ Minimum value when $x = 2$

Goldsmith should use $x = 2$ to minimise the amount of gold plating used.

## Question 9

(a) Roots of parabola are $x = 0$ and $x = 4$

$\therefore$ Equation is of the form

$$y = kx(x-4)$$

Since the point $(2, 4)$ lies on the parabola

$$4 = k \times 2 \times (2-4)$$
$$= k \times 2 \times (-2)$$

so $-4k = 4$

$$k = -1$$

$\therefore$ Equation of the parabola is

$$y = -x(x-4) = -x^2 + 4x$$

So $y = 4x - x^2$

(b) Shaded area $= \int_2^k 4x - x^2 \, dx$

$$= \left[2x^2 - \tfrac{1}{3}x^3\right]_2^k$$

$$= \left(2k^2 - \tfrac{1}{3}k^3\right) - \left(2(2^2) - \tfrac{1}{3}(2^3)\right)$$

$$= 2k^2 - \tfrac{1}{3}k^3 - \left(2^3 - \tfrac{1}{3}(2^3)\right)$$

$$= 2k^2 - \tfrac{1}{3}k^3 - \tfrac{2}{3} \cdot 2^3$$

$$= 2k^2 - \tfrac{1}{3}k^3 - \tfrac{16}{3}$$

$\therefore A = -\frac{1}{3}k^3 + 2k^2 - \frac{16}{3}$ as required.

## Question 10

$3\cos 2x^\circ + \cos x^\circ = -1 \qquad 0 \leq x \leq 360.$

$\therefore \quad 3\cos 2x + \cos x + 1 = 0$

$3(2\cos^2 x - 1) + \cos x + 1 = 0$

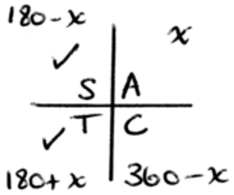
$6\cos^2 x - 3 + \cos x + 1 = 0$

$6\cos^2 x + \cos x - 2 = 0$

$(3\cos x + 2)(2\cos x - 1) = 0.$

$\therefore \quad 3\cos x + 2 = 0$ or $2\cos x - 1 = 0$

$\cos x = -\frac{2}{3}$ or $\cos x = \frac{1}{2}$



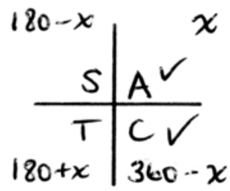

$\cos x = \frac{2}{3}$

$x = \cos^{-1}\left(\frac{2}{3}\right)$

$= 48{\cdot}2^\circ$ (to 1dp)

$\therefore x = 180 - 48{\cdot}2$ or $180 + 48{\cdot}2$

$= 131{\cdot}8$ or $228{\cdot}2$ (to 1dp)




$x = \cos^{-1}\left(\frac{1}{2}\right)$

$= 60$ or $360 - 60$

$= 60$ or $300$

$\therefore$ Solutions are $\{60,\ 131{\cdot}8,\ 228{\cdot}2,\ 300\}$

## Question 11

(a) gradient of line $m = \dfrac{1{\cdot}8 - 0}{0 - (-3)} = \dfrac{1{\cdot}8}{3} = 0{\cdot}6$

$y$-intercept is $1{\cdot}8$

$\therefore$ Equation of line is

$P = 0{\cdot}6Q + 1{\cdot}8.$

(b) Method 1. $p = aq^b$.

$$\log_e p = \log_e (aq^b)$$
$$= \log_e a + \log_e q^b$$
$$= \log_e a + b\log_e q$$
$$= b\log_e q + \log_e a$$

Comparing this with above line

$$P = 0.6Q + 1.8$$

so $\log_e p = 0.6\log_e q + 1.8$

$\therefore$ $b = 0.6$ and $\log_e a = 1.8$

$$a = e^{1.8} = 6.05 \text{ (to 2dp)}$$

Method 2 $P = 0.6Q + 1.8$

$$\log_e p = 0.6\log_e q + 1.8$$
$$= \log_e q^{0.6} + 1.8\log_e e$$
$$= \log_e q^{0.6} + \log_e e^{1.8}$$
$$= \log_e e^{1.8} q^{0.6}$$

$\therefore$ $p = e^{1.8} q^{0.6}$ so $a = e^{1.8} = 6.05$ and $b = 0.6$