Advanced Higher 2006

Marks

Answer all the questions.

1. Calculate the inverse of the matrix $\begin{pmatrix} 2 & x \\ -1 & 3 \end{pmatrix}$.

For what value of x is this matrix singular? **4**

2. Differentiate, simplifying your answers:

(*a*) $2\tan^{-1}\sqrt{1+x}$, where $x > -1$; **3**

(*b*) $\dfrac{1+\ln x}{3x}$, where $x > 0$. **3**

3. Express the complex number $z = -i + \dfrac{1}{1-i}$ in the form $z = x + iy$, stating the values of x and y. **3**

Find the modulus and argument of z and plot z and $\bar{z}$ on an Argand diagram. **4**

4. Given $xy - x = 4$, use implicit differentiation to obtain $\dfrac{dy}{dx}$ in terms of x and y. **2**

Hence obtain $\dfrac{d^2y}{dx^2}$ in terms of x and y. **3**

5. Obtain algebraically the fixed point of the iterative scheme given by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n^2}\right), \qquad n = 0, 1, 2, \ldots .$$ **3**

6. Find $\displaystyle\int \frac{12x^3 - 6x}{x^4 - x^2 + 1}\,dx$. **3**

7. For all natural numbers n, prove whether the following results are true or false.

(*a*) $n^3 - n$ is always divisible by 6.

(*b*) $n^3 + n + 5$ is always prime. **5**

8. Solve the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0,$$

given that when $x = 0$, $y = 0$ and $\dfrac{dy}{dx} = 2$. **6**

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9. Use Gaussian elimination to obtain solutions of the equations

$$\begin{aligned} 2x - y + 2z &= 1 \\ x + y - 2z &= 2 \\ x - 2y + 4z &= -1. \end{aligned}$$

5

10. The amount x micrograms of an impurity removed per kg of a substance by a chemical process depends on the temperature T °C as follows:

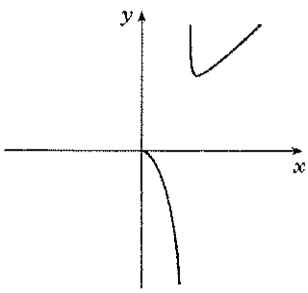
$$x = T^3 - 90T^2 + 2400T, \quad 10 \le T \le 60.$$

At what temperature in the given range should the process be carried out to remove as much impurity per kg as possible? 4

11. Show that $1 + \cot^2\theta = \operatorname{cosec}^2\theta$, where $0 < \theta < \frac{\pi}{2}$. 1

By expressing $y = \cot^{-1}x$ as $x = \cot y$, obtain $\frac{dy}{dx}$ in terms of x. 3

12.



The diagram shows part of the graph of a function f which satisfies the following conditions:

(i) f is an even function;
(ii) two of the asymptotes of the graph $y = f(x)$ are $y = x$ and $x = 1$.

Copy the diagram and complete the graph. Write down equations for the other two asymptotes. 3

13. The square matrices A and B are such that $AB = BA$. Prove by induction that $A^nB = BA^n$ for all integers $n \ge 1$. 5

[Turn over for Questions 14 to 17 on *Page four*

Marks

14. (*a*) Determine whether $f(x) = x^2 \sin x$ is odd, even or neither. Justify your answer. **3**

(*b*) Use integration by parts to find $\int x^2 \sin x \, dx$. **4**

(*c*) Hence find the area bounded by $y = x^2 \sin x$, the lines $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$ and the x-axis. **3**

15. Obtain an equation for the plane passing through the point $P(1, 1, 0)$ which is perpendicular to the line L given by

$$\frac{x+1}{2} = \frac{y-2}{1} = \frac{z}{-1}.$$ **3**

Find the coordinates of the point Q where the plane and L intersect. **4**

Hence, or otherwise, obtain the shortest distance from P to L and explain why this is the shortest distance. **2, 1**

16. The first three terms of a geometric sequence are

$$\frac{x(x+1)}{(x-2)}, \frac{x(x+1)^2}{(x-2)^2} \text{ and } \frac{x(x+1)^3}{(x-2)^3}, \text{ where } x < 2.$$

(*a*) Obtain expressions for the common ratio and the nth term of the sequence. **3**

(*b*) Find an expression for the sum of the first n terms of the sequence. **3**

(*c*) Obtain the range of values of x for which the sequence has a sum to infinity and find an expression for the sum to infinity. **4**

17. (*a*) Show that $\int \sin^2 x \cos^2 x \, dx = \int \cos^2 x \, dx - \int \cos^4 x \, dx$. **1**

(*b*) By writing $\cos^4 x = \cos x \cos^3 x$ and using integration by parts, show that

$$\int_0^{\pi/4} \cos^4 x \, dx = \frac{1}{4} + 3\int_0^{\pi/4} \sin^2 x \cos^2 x \, dx.$$ **3**

(*c*) Show that $\int_0^{\pi/4} \cos^2 x \, dx = \frac{\pi+2}{8}$. **3**

(*d*) Hence, using the above results, show that

$$\int_0^{\pi/4} \cos^4 x \, dx = \frac{3\pi+8}{32}.$$ **3**

[*END OF QUESTION PAPER*]