2005 Advanced Higher

1. (*a*) Given $f(x) = x^3 \tan 2x$, where $0 < x < \frac{\pi}{4}$, obtain $f'(x)$. 3

(*b*) For $y = \frac{1+x^2}{1+x}$, where $x \neq -1$, determine $\frac{dy}{dx}$ in simplified form. 3

2. Given the equation $2y^2 - 2xy - 4y + x^2 = 0$ of a curve, obtain the x-coordinate of each point at which the curve has a horizontal tangent. 4

3. Write down the Maclaurin expansion of e^x as far as the term in x^4. 2

Deduce the Maclaurin expansion of e^{x^2} as far as the term in x^4. 1

Hence, or otherwise, find the Maclaurin expansion of e^{x+x^2} as far as the term in x^4. 3

4. The sum, $S(n)$, of the first n terms of a sequence, $u_1, u_2, u_3, \ldots$ is given by $S(n) = 8n - n^2$, $n \geq 1$.

Calculate the values of u_1, u_2, u_3 and state what type of sequence it is. 3

Obtain a formula for u_n in terms of n, simplifying your answer. 2

5. Use the substitution $u = 1 + x$ to evaluate $\int_0^3 \frac{x}{\sqrt{1+x}}\,dx$. 5

6. Use Gaussian elimination to solve the system of equations below when $\lambda \neq 2$:

$$\begin{aligned} x + y + 2z &= 1 \\ 2x + \lambda y + z &= 0 \\ 3x + 3y + 9z &= 5. \end{aligned}$$ 4

Explain what happens when $\lambda = 2$. 2

7. Given the matrix $A = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$, show that $A^2 + A = kI$ for some constant k, where I is the 3×3 unit matrix. 4

Obtain the values of p and q for which $A^{-1} = pA + qI$. 2

8. The equations of two planes are $x - 4y + 2z = 1$ and $x - y - z = -5$. By letting $z = t$, or otherwise, obtain parametric equations for the line of intersection of the planes. 4

Show that this line lies in the plane with equation

$$x + 2y - 4z = -11.$$ 1

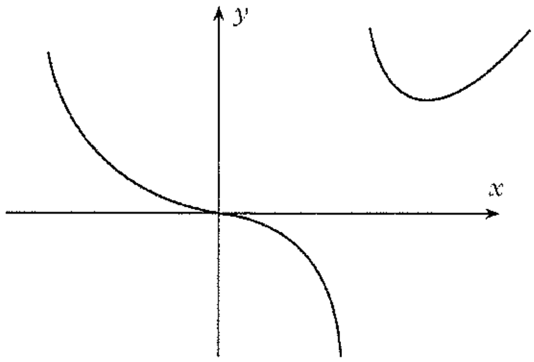
9. Given the equation $z + 2i\bar{z} = 8 + 7i$, express z in the form $a + ib$. **4**

10. Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$ **5**

State the value of $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$. **1**

11. The diagram shows part of the graph of $y = \frac{x^3}{x-2}$, $x \neq 2$.



(a) Write down the equation of the vertical asymptote. **1**

(b) Find the coordinates of the stationary points of the graph of $y = \frac{x^3}{x-2}$. **4**

(c) Write down the coordinates of the stationary points of the graph of $y = \left|\frac{x^3}{x-2}\right| + 1$. **2**

12. Let $z = \cos\theta + i\sin\theta$.

(a) Use the binomial expansion to express z^4 in the form $u + iv$, where u and v are expressions involving $\sin\theta$ and $\cos\theta$. **3**

(b) Use de Moivre's theorem to write down a second expression for z^4. **1**

(c) Using the results of (a) and (b), show that

$$\frac{\cos 4\theta}{\cos^2\theta} = p\cos^2\theta + q\sec^2\theta + r, \text{ where } -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

stating the values of p, q and r. **6**

13. Express $\dfrac{1}{x^3 + x}$ in partial fractions. **4**

Obtain a formula for $I(k)$, where $I(k) = \displaystyle\int_1^k \frac{1}{x^3 + x}\,dx$, expressing it in the form $\ln\left(\dfrac{a}{b}\right)$, where a and b depend on k. **4**

Write down an expression for $e^{I(k)}$ and obtain the value of $\lim\limits_{k \to \infty} e^{I(k)}$. **2**

14. Obtain the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 20 \sin x.$$ **7**

Hence find the particular solution for which $y = 0$ and $\dfrac{dy}{dx} = 0$ when $x = 0$. **3**

15. (*a*) Given $f(x) = \sqrt{\sin x}$, where $0 < x < \pi$, obtain $f'(x)$. **1**

(*b*) If, in general, $f(x) = \sqrt{g(x)}$, where $g(x) > 0$, show that $f'(x) = \dfrac{g'(x)}{k\sqrt{g(x)}}$, stating the value of k. **2**

Hence, or otherwise, find $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$. **3**

(*c*) Use integration by parts and the result of (*b*) to evaluate

$$\int_0^{1/2} \sin^{-1} x\,dx.$$ **4**

[*END OF QUESTION PAPER*]