2004 Advanced Higher

1. (*a*) Given $f(x) = \cos^2 x\, e^{\tan x}$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, obtain $f'(x)$ and evaluate $f'(\frac{\pi}{4})$. 3,1

(*b*) Differentiate $g(x) = \dfrac{\tan^{-1} 2x}{1+4x^2}$. 3

2. Obtain the binomial expansion of $(a^2 - 3)^4$. 3

3. A curve is defined by the equations

$$x = 5\cos\theta, \qquad y = 5\sin\theta, \qquad (0 \le \theta < 2\pi).$$

Use parametric differentiation to find $\dfrac{dy}{dx}$ in terms of θ. 2

Find the equation of the tangent to the curve at the point where $\theta = \dfrac{\pi}{4}$. 3

4. Given $z = 1 + 2i$, express $z^2(z + 3)$ in the form $a + ib$. 2

Hence, or otherwise, verify that $1 + 2i$ is a root of the equation

$$z^3 + 3z^2 - 5z + 25 = 0.$$ 2

Obtain the other roots of this equation. 2

5. Express $\dfrac{1}{x^2 - x - 6}$ in partial fractions. 2

Evaluate $\displaystyle\int_0^1 \frac{1}{x^2 - x - 6}\,dx$. 4

6. Write down the 2×2 matrix M_1 associated with an anti-clockwise rotation of $\frac{\pi}{2}$ radians about the origin. 2

Write down the matrix M_2 associated with reflection in the x-axis. 1

Evaluate $M_2 M_1$ and describe geometrically the effect of the transformation represented by $M_2 M_1$. 2

7. Obtain the first three non-zero terms in the Maclaurin expansion of $f(x) = e^x \sin x$. 5

8. Use the Euclidean algorithm to show that $(231, 17) = 1$ where (a, b) denotes the highest common factor of a and b.

Hence find integers x and y such that $231x + 17y = 1$. 4

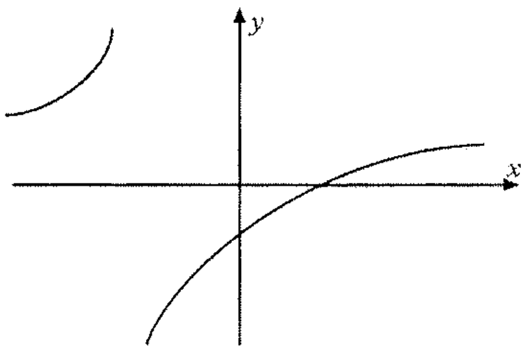
9. Use the substitution $x = (u - 1)^2$ to obtain $\displaystyle\int \frac{1}{(1+\sqrt{x})^3}\,dx$. 5

10. Determine whether the function $f(x) = x^4 \sin 2x$ is odd, even or neither.
Justify your answer. **3**

11. A solid is formed by rotating the curve $y = e^{-2x}$ between $x = 0$ and $x = 1$ through $360°$ about the x-axis. Calculate the volume of the solid that is formed. **5**

12. Prove by induction that $\frac{d^n}{dx^n}(xe^x) = (x + n)e^x$ for all integers $n \geq 1$. **5**

13. The function f is defined by $f(x) = \frac{x-3}{x+2}$, $x \neq -2$, and the diagram shows part of its graph.



(*a*) Obtain algebraically the asymptotes of the graph of f. **3**

(*b*) Prove that f has no stationary values. **2**

(*c*) Does the graph of f have any points of inflexion? Justify your answer. **2**

(*d*) Sketch the graph of the inverse function, f^{-1}. State the asymptotes and domain of f^{-1}. **3**

14. (*a*) Find an equation of the plane π_1 containing the points $A(1, 0, 3)$, $B(0, 2, -1)$ and $C(1, 1, 0)$. **4**

Calculate the size of the acute angle between π_1 and the plane π_2 with equation $x + y - z = 0$. **3**

(*b*) Find the point of intersection of plane π_2 and the line

$$\frac{x-11}{4} = \frac{y-15}{5} = \frac{z-12}{2}.$$ **3**

15. (*a*) A mathematical biologist believes that the differential equation $x\frac{dy}{dx} - 3y = x^4$ models a process. Find the general solution of the differential equation. **5**

Given that $y = 2$ when $x = 1$, find the particular solution, expressing y in terms of x. **2**

(*b*) The biologist subsequently decides that a better model is given by the equation $y\frac{dy}{dx} - 3x = x^4$.

Given that $y = 2$ when $x = 1$, obtain y in terms of x. **4**

16. (*a*) Obtain the sum of the series $8 + 11 + 14 + \ldots + 56$. **2**

(*b*) A geometric sequence of positive terms has first term 2, and the sum of the first three terms is 266. Calculate the common ratio. **3**

(*c*) An arithmetic sequence, A, has first term a and common difference 2, and a geometric sequence, B, has first term a and common ratio 2. The first four terms of each sequence have the same sum. Obtain the value of a. **3**

Obtain the smallest value of n such that the sum to n terms for sequence B is more than **twice** the sum to n terms for sequence A. **2**

[END OF QUESTION PAPER]