

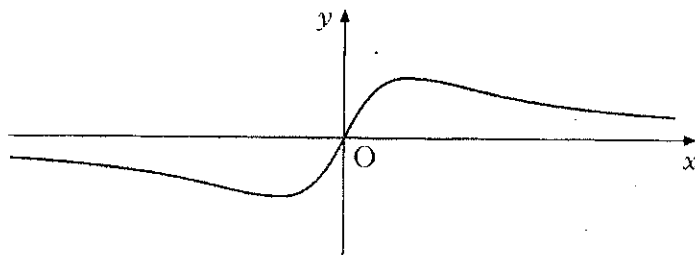
# 2003 Advanced Higher

1. (a) Given  $f(x) = x(1+x)^{10}$ , obtain  $f'(x)$  and simplify your answer. 3  
 (b) Given  $y = 3^x$ , use logarithmic differentiation to obtain  $\frac{dy}{dx}$  in terms of  $x$ . 3
2. Given that  $u_k = 11 - 2k$ , ( $k \geq 1$ ), obtain a formula for  $S_n = \sum_{k=1}^n u_k$ . 3  
 Find the values of  $n$  for which  $S_n = 21$ . 2
3. The equation  $y^3 + 3xy = 3x^2 - 5$  defines a curve passing through the point  $A(2, 1)$ . Obtain an equation for the tangent to the curve at  $A$ . 4
4. Identify the locus in the complex plane given by  $|z + i| = 2$ . 3
5. Use the substitution  $x = 1 + \sin \theta$  to evaluate  $\int_0^{\pi/2} \frac{\cos \theta}{(1 + \sin \theta)^3} d\theta$ . 5
6. Use elementary row operations to reduce the following system of equations to upper triangular form

$$\begin{array}{rcl} x + y + 3z & = & 1 \\ 3x + ay + z & = & 1 \\ x + y + z & = & -1. \end{array}$$
2

Hence express  $x$ ,  $y$  and  $z$  in terms of the parameter  $a$ . 2  
 Explain what happens when  $a = 3$ . 2

7.



The diagram shows the shape of the graph of  $y = \frac{x}{1+x^2}$ . Obtain the stationary points of the graph. 4

Sketch the graph of  $y = \left| \frac{x}{1+x^2} \right|$  and identify its three critical points. 3

8. Given that  $p(n) = n^2 + n$ , where  $n$  is a positive integer, consider the statements:

- A  $p(n)$  is always even
- B  $p(n)$  is always a multiple of 3.

For each statement, prove it if it is true or, otherwise, disprove it.

4

9. Given that  $w = \cos \theta + i \sin \theta$ , show that  $\frac{1}{w} = \cos \theta - i \sin \theta$ .

1

Use de Moivre's theorem to prove  $w^k + w^{-k} = 2 \cos k\theta$ , where  $k$  is a natural number.

3

Expand  $(w + w^{-1})^4$  by the binomial theorem and hence show that

$$\cos^4 \theta = \frac{1}{8} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8}.$$

5

10. Define  $I_n = \int_0^1 x^n e^{-x} dx$  for  $n \geq 1$ .

(a) Use integration by parts to obtain the value of  $I_1 = \int_0^1 x e^{-x} dx$ .

3

(b) Similarly, show that  $I_n = nI_{n-1} - e^{-1}$  for  $n \geq 2$ .

4

(c) Evaluate  $I_3$ .

3

11. The volume  $V(t)$  of a cell at time  $t$  changes according to the law

$$\frac{dV}{dt} = V(10 - V) \quad \text{for } 0 < V < 10.$$

Show that

$$\frac{1}{10} \ln V - \frac{1}{10} \ln (10 - V) = t + C$$

for some constant  $C$ .

4

Given that  $V(0) = 5$ , show that

$$V(t) = \frac{10e^{10t}}{1 + e^{10t}}.$$

3

Obtain the limiting value of  $V(t)$  as  $t \rightarrow \infty$ .

2

12. Find the point of intersection of the line

$$\frac{x-3}{4} = \frac{y-2}{-1} = \frac{z+1}{2}$$

and the plane with equation  $2x + y - z = 4$ .

4

13. The matrix  $A$  is such that  $A^2 = 4A - 3I$  where  $I$  is the corresponding identity matrix. Find integers  $p$  and  $q$  such that

$$A^4 = pA + qI.$$

4

14. A recurrence relation is defined by the formula

$$x_{n+1} = \frac{1}{2} \left\{ x_n + \frac{7}{x_n} \right\}.$$

Find the fixed points of this recurrence relation.

3

15. Obtain the Maclaurin series for  $f(x) = \sin^2 x$  up to the term in  $x^4$ .  
Hence write down a series for  $\cos^2 x$  up to the term in  $x^4$ .

4

1

16. (a) Prove by induction that for all natural numbers  $n \geq 1$

$$\sum_{r=1}^n 3(r^2 - r) = (n-1)n(n+1).$$

4

- (b) Hence evaluate  $\sum_{r=1}^{40} 3(r^2 - r)$ .

2

17. Solve the differential equation

$$\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = e^x,$$

given that  $y = 2$  and  $\frac{dy}{dx} = 1$ , when  $x = 0$ .

10

[END OF QUESTION PAPER]