Advanced Higher Mathematics 2002

1. $x = 2, y = -3, z = 1$

2. $i^4 + 4i^3 + 3i^2 + 4i + 2 = 1 - 4i - 3 + 4i + 2 = 0$
   Roots are $i, -i, -2 + \sqrt{2}, -2 - \sqrt{2}$

3. At A, $x = -1$. So $t^2 + t - 1 = -1$.
   Hence $t = 0$ or $t = -1$.
   When $t = 0$, $y = 2$.
   When $t = -1$, $y = 5$, so A is on the curve.
   $\frac{dy}{dx} = \frac{4t-1}{2t+1}$ hence when $t = -1$, $\frac{dy}{dx} = 5$
   Tangent is $y = 5x + 10$

4. (*a*) $f'(x) = \frac{1}{2\sqrt{x}}e^{-x}(1 - 2x)$
   (*b*) $y = (x + 1)^2(x + 2)^{-4}$
   $\log y = 2\log(x + 1) - 4\log(x + 2)$
   $\frac{1}{y}\frac{dy}{dx} = \frac{2}{x+1} - \frac{4}{x+2}$
   $\frac{dy}{dx} = \left(\frac{2}{x+1} - \frac{4}{x+2}\right)y$

   $a = 2, b = -4$

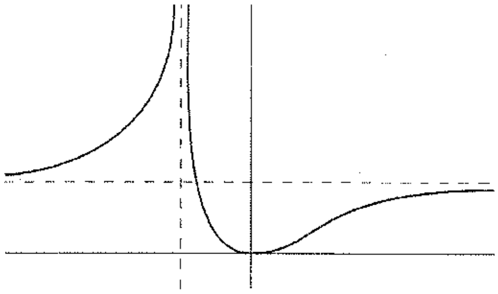
5. $\int_0^1 \ln(1 + x)\,dx = 2\ln 2 - 1$

6. $\int \frac{dx}{x^2 + 4x + 8} = \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + c$

7. When $n = 1$, $4^n - 1 = 4 - 1 = 3$, so true when $n = 1$.
   Assume $4^k - 1$ is divisible by 3.
   Consider $4^{k+1} - 1$.
   $4^{k+1} - 1 = 4 \times 4^k - 1$
   $= (3 + 1)4^k - 1$
   $= 3 \times 4^k + (4^k - 1)$
   Since both terms are divisible by 3, the result is true for $k + 1$.
   Thus, since true for $n = 1$, $4^n - 1$ is divisible by 3 for all $n \geqslant 1$.

8. $A = 1, B = -2, C = 1$
   (i) vertical asymptote: $x = -1$
   horizontal asymptote: $y = 1$
   (ii) (0,0) is a minimum because at (0, 0) $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

8. (iii)



9. (*a*) $\frac{dy}{dx} = \frac{y}{x}$ Hence $y = 2x$
   (*b*) $\frac{dx}{dt} = -x^2(2x) = -2x^3$
   $x = \frac{1}{\sqrt{4t+1}}$

10. $S_n(1) = 1 + 2 + 3 + \ldots + n$
    $= \frac{1}{2}n(n + 1)$
    $(1 - x)S_n(x) = S_n(x) - xS_n(x)$
    $= 1 + 2x + 3x^2 + \ldots + nx^{n-1}$
    $-(x + 2x^2 + 3x^3 + \ldots + nx^n)$
    $= 1 + x + x^2 + \ldots + nx^{n-1} - nx^n)$
    $= \frac{1-x^n}{1-x} - nx^n$
    Thus $S_n(x) = \frac{1-x^n}{(1-x)^2} - \frac{nx^n}{(1-x)}$
    $\lim_{n\to\infty}\left\{\frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \ldots + \frac{n}{3^{n-1}} + \frac{3}{2}\cdot\frac{n}{3^n}\right\} = \frac{5}{4}$

11. (*a*) An equation is $-x + 5y - 2z = 4$
    (*b*) $\cos^{-1}\frac{7}{6\sqrt{5}} \approx 58{\cdot}6°$

12. $A^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix}$ When $n = 1$.

    $\text{RHS} = \begin{pmatrix} 1+1 & 1 \\ -1 & 1-1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = A$
    So true for $n = 1$.

    $A^k = \begin{pmatrix} k+1 & k \\ -k & 1-k \end{pmatrix}$

    Consider A^{k+1}
    $A^{k+1} = A \times A^n$
    $= \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} k+1 & k \\ -k & 1-k \end{pmatrix}$
    $= \begin{pmatrix} k+2 & k+1 \\ -(k+1) & -k \end{pmatrix}$
    $= \begin{pmatrix} (k+1)+1 & (k+1) \\ -(k+1) & 1-(k+1) \end{pmatrix}$
    So if true for k, true for $k + 1$.
    Since true for $n = 1$, by induction, true for all $n \geqslant 1$.

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13. $\ln(\cos x) = -\frac{x^2}{2} - \frac{x^4}{12} + \ldots$

14. $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $B = \begin{pmatrix} \cos30^\circ & -\sin30^\circ \\ \sin30^\circ & \cos30^\circ \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$

$BA\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3} + y \\ x - \sqrt{3}y \end{pmatrix}$

i.e. $(x,y) \to \frac{1}{2}(\sqrt{3}x + y, x - \sqrt{3}y)$

so $k = \sqrt{3}$

15. General solution:

$y(x) = e^{-x}(A\cos2x + B\sin2x) + \frac{2}{5}(2\cos x + \sin x)$

Particular solution:

$y(x) = \frac{e^{-x}}{10}(-8\cos2x - \sin2x) + \frac{2}{5}(2\cos x + \sin2x)$