Question A1

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 2 & -1 & 3 & 4 \\ 1 & 0 & 2 & 20 \end{array}\right)\begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array}$$

$$\begin{array}{l} r_1' = r_1 \\ r_2' = r_2 - 2r_1 \\ r_3' = r_3 - r_1 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -16 \\ 0 & -1 & 1 & 10 \end{array}\right)$$

$$\begin{array}{l} r_1'' = r_1' \\ r_2'' = r_2' \\ r_3'' = 3r_3' - r_2' \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -16 \\ 0 & 0 & 2 & 46 \end{array}\right)$$

Using back substitution…

$$2z = 46$$
$$z = 23$$

So
$$-3y + x = 16$$
$$-3y + 23 = 16$$
$$3y = 39$$
$$y = 13$$

And
$$x + y + z = 10$$
$$x + 13 + 23 = 10$$
$$x + 36 = 10$$
$$x = -26$$

Question A2

(a) $f(x) = (2+x)\tan^{-1}\sqrt{x-1}$

Using the product rule…

$$f'(x) = 1.\tan^{-1}\sqrt{x-1} + (2+x).\frac{1}{1+\left(\sqrt{x-1}\right)^2}.\tfrac{d}{dx}\left(\sqrt{x-1}\right)$$

$$= \tan^{-1}\sqrt{x-1} + (2+x).\frac{1}{1+x-1}.\tfrac{1}{2}(x-1)^{-\frac{1}{2}}.1$$

$$= \tan^{-1}\sqrt{x-1} + \frac{2+x}{x}.\frac{1}{2\sqrt{x-1}}$$

$$= \tan^{-1}\sqrt{x-1} + \frac{2+x}{2x\sqrt{x-1}}$$

(b) $g(x) = e^{\cot 2x}$

Method 1

$$g'(x) = -\text{cosec}^2(2x).2.e^{\cot 2x}$$

$$= -2e^{\cot 2x}\,\text{cosec}^2\,2x$$

Method 2

$$g'(x) = e^{\cot 2x}.\tfrac{d}{dx}\left(\frac{\cos 2x}{\sin 2x}\right)$$

$$= e^{\cot 2x}.\frac{-2\sin^2 2x - 2\cos^2 2x}{\sin^2 2x} \quad \text{using the quotient rule}$$

$$= e^{\cot 2x}\left(-\frac{2\sin^2 2x}{\sin^2 2x} - \frac{2\cos^2 2x}{\sin^2 2x}\right)$$

$$= e^{\cot 2x}\left(-2 - \frac{2\cos^2 2x}{\sin^2 2x}\right)$$

$$= -2e^{\cot 2x}\left(1+\cot^2 2x\right)$$

Question A3

$$\int_0^{\frac{\pi}{4}} 2x\sin 4x\ dx$$

Using integration by parts...

$$=\left[2x.-\tfrac{1}{4}\cos 4x\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}-\tfrac{1}{2}\cos 4x\ dx$$

$$=\left[-\tfrac{1}{2}x\cos 4x\right]_0^{\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}}\tfrac{1}{2}\cos 4x\ dx$$

$$=\left[-\tfrac{1}{2}x\cos 4x\right]_0^{\frac{\pi}{4}}+\left[\tfrac{1}{2}.\tfrac{1}{4}\sin(4x)\right]_0^{\frac{\pi}{4}}$$

$$=\left[-\tfrac{1}{2}x\cos 4x+\tfrac{1}{8}\sin 4x\right]_0^{\frac{\pi}{4}}$$

$$=\left(-\tfrac{\pi}{8}\cos\pi+\tfrac{1}{8}\sin\pi\right)-\left(0+\tfrac{1}{8}\sin\pi\right)$$

$$=-\tfrac{\pi}{8}.-1$$

$$=\tfrac{\pi}{8}$$

Question A4

$$2+5+8+\cdots+(3n-1)=\tfrac{1}{2}n(3n-1)$$

When $n=1$, $\text{LHS}=2$

$\text{RHS}=\tfrac{1}{2}.1.4=2$

Since $\text{RHS}=\text{LHS}$, the result is true for $n=1$

Assume true for $n=k$

$$2+5+8+\cdots+(3k-1)=\tfrac{1}{2}k(3k+1)$$

When $n=k+1$

$$\begin{aligned}2+5+8+\cdots+(3k-1)+\left(3(3k+1)-1\right)&=\tfrac{1}{2}k(3k+1)+\left(3(k+1)-1\right)\\&=\tfrac{1}{2}k(3k+1)+(3k+2)\\&=\tfrac{3}{2}k^2+\tfrac{1}{2}k+3k+2\\&=\tfrac{3}{2}k^2+\tfrac{7}{2}k+2\\&=\tfrac{1}{2}\left(3k^2+7k+4\right)\\&=\tfrac{1}{2}(k+1)(3k+4)\\&=\tfrac{1}{2}(k+1)(3k+3+1)\\&=\tfrac{1}{2}(k+1)\left(3(k+1)+1\right)\end{aligned}$$

Since true for $n=1$ and $n=k+1$, the by the Principle of Mathematical Induction, the result is true for all positive integers.

Question A5

(a) $\dfrac{x}{x^2-1}=\dfrac{x}{(x+1)(x-1)}$

$$=\frac{A}{x-1}+\frac{B}{x+1}$$

$$=\frac{A(x+1)+B(x-1)}{(x+1)(x-1)}$$

Equating numerators...

$$A(x-1)+B(x+1)=x$$

When $x=-1$: $\quad -2A=-1 \quad \Rightarrow \quad A=\tfrac{1}{2}$

When $x=1$: $\quad 2B=1 \quad \Rightarrow \quad B=\tfrac{1}{2}$

$$\therefore \quad \frac{x}{x^2-1}=\frac{1}{2(x+1)}+\frac{1}{2(x-1)}$$

(b) $\displaystyle\int \frac{x^3}{x^2-1}\,dx$

$$=\int x+\frac{x}{x^2-1}\,dx$$

$$=\int x+\frac{\frac{1}{2}}{x+1}+\frac{\frac{1}{2}}{x-1}\,dx$$

$$=\int x\,dx+\tfrac{1}{2}\int\frac{1}{x+1}\,dx+\tfrac{1}{2}\int\frac{1}{x+1}\,dx$$

$$=\tfrac{1}{2}x^2+\tfrac{1}{2}\ln|x+1|+\tfrac{1}{2}\ln|x-1|+c$$

$$=\tfrac{1}{2}x^2+\tfrac{1}{2}\ln\left|x^2-1\right|+c$$

Question A6

$$\left(x^2-\frac{2}{x}\right)^4$$

$$=\binom{4}{0}(x^2)^4+\binom{4}{1}(x^2)^3\left(-\frac{2}{x}\right)+\binom{4}{2}(x^2)^2\left(-\frac{2}{x}\right)^2+\binom{4}{3}(x^2)\left(-\frac{2}{x}\right)^3+\binom{4}{4}\left(-\frac{2}{x}\right)^4$$

$$=x^8+4x^6\left(-\frac{2}{x}\right)+6x^4\left(\frac{4}{x^2}\right)+4x^2\left(-\frac{8}{x^3}\right)+\frac{16}{x^4}$$

$$=x^8-8x^5+24x^2-\frac{32}{x}+\frac{16}{x^4}$$

Question A7

(a)
$$xy + y^2 = 2$$
$$\frac{d}{dx}\left(xy + y^2\right) = \frac{d}{dx}(2)$$
$$y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$
$$(x+2y)\frac{dy}{dx} = -y$$
$$\frac{dy}{dx} = -\frac{y}{x+2y}$$

(b) At $(1,1)$, $m_{\text{tangent}} = \frac{dy}{dx} = -\frac{1}{1+2} = -\frac{1}{3}$

Therefore the equation of the tangent is...
$$y - 1 = -\tfrac{1}{3}(x-1)$$
$$3y - 3 = -x + 1$$
$$x + 3y - 4 = 0$$

Question A8

(a)
$$f(x) = \frac{x^2+6x+12}{x+2}$$
$$= x + 4 + \frac{4}{x+2}$$

Therefore $a = 1$ and $b = 4$

(b) Vertical asymptote: $x = -2$

Non-vertical asymptote: $y = x + 4$

(c) $f'(x) = 1 - 4(x+2)^{-2}$

$$= 1 - \frac{4}{(x+2)^2}$$

Stationary points occur where $f'(x) = 0\ldots$

$$1 - \frac{4}{(x+2)^2} = 0$$

$$1 = \frac{4}{(x+2)^2}$$

$$(x+2)^2 = 4$$

$$x + 2 = \pm 2$$

$$x = -2 \pm 2$$

$$= 0 \text{ or } -4$$

When $x = 0$ then $f(0) = \frac{12}{2} = 6$

When $x = -4$ then $f(-4) = \dfrac{16 - 24 + 12}{-2} = -\frac{4}{2} = -2$

Nature

Method 1 – Second derivative

$$f''(x) = \frac{8}{(x+2)^3}$$

When $x = 0, f''(0) = \dfrac{8}{2^3} = 1 > 0 \Rightarrow$ Local minimum at $(0, 6)$

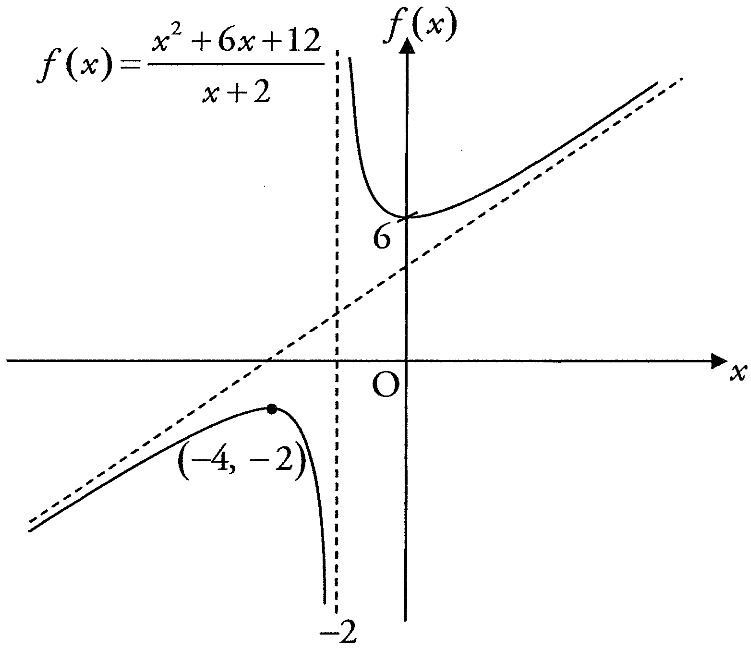
When $x = -4, f''(-4) = \dfrac{8}{(-2)^3} = -1 < 0 \Rightarrow$ Local maximum at $(-4, -2)$

Method 2 – Table of values

x	$\rightarrow$	-4	$\rightarrow$	0	$\rightarrow$
x	$-$	$-$	$-$	0	$+$
$x+4$	$-$	0	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$	0	$+$
Graph	/	—	\	_	/

Hence $(0, 6)$ is a local minimum and $(-4, -2)$ is a local maximum.

(d)



(e) $-2<k<6$

Question A9

(a) $\theta=\pi$

(b) Let $z=\cos\theta+i\sin\theta$

Then $z^3=(\cos\theta+i\sin\theta)^3=\cos3\theta+i\sin3\theta$

$\therefore\ z^3+1=0$

$z^3=-1$

And so $\cos3\theta+i\sin3\theta=-1$, ie $3\theta=\pi$ or 3π or 5π

$\theta=\frac{\pi}{3}$ or π or $\frac{5\pi}{3}$

Therefore the solutions are $\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$\cos\pi+i\sin\pi=-1$

$\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}=\frac{1}{2}-\frac{\sqrt{3}}{2}i$

Thus $z_1^2=\left[\frac{1}{2}\left(1+\sqrt{3}i\right)\right]^2=\frac{1}{4}\left(1+2\sqrt{3}i+3i^2\right)=\frac{1}{4}\left(-2+2\sqrt{3}i\right)=-\frac{1}{2}\left(1-\sqrt{3}i\right)=-z_2$

$z_2^2=\left[\frac{1}{2}\left(1-\sqrt{3}i\right)\right]^2=\frac{1}{4}\left(1-2\sqrt{3}i+3i^2\right)=\frac{1}{4}\left(-2+2\sqrt{3}i\right)=-\frac{1}{2}\left(1+\sqrt{3}i\right)=-z_1$

(c) Points form an equilateral triangle.

Points are equally spaced around a circle with radius 1 unit.

Question A10

(a)
$$\frac{dM}{dt} = kM$$
$$\Rightarrow \int \frac{1}{M}\, dM = \int k\, dt$$
$$\ln|M| = kt + c$$
$$M = e^{kt+c}$$
$$= e^{c}.e^{kt}$$
$$= Ae^{kt}$$

When $t = 0$, $M = M_0$ so $M_0 = Ae^0 = A$.

Hence $M = M_0 e^{kt}$

(b) When $t = 30$, $M = \frac{1}{2}M_0$:

$$\tfrac{1}{2}M_0 = M_0 e^{30k}$$
$$\Rightarrow e^{30k} = \tfrac{1}{2}$$
$$30k = \ln\left(\tfrac{1}{2}\right)$$
$$k = \tfrac{1}{30}\ln\left(\tfrac{1}{2}\right)$$
$$= -0{\cdot}023104906\ldots$$

Therefore $k = -0{\cdot}0231$ to 3 significant figures.

(c) When $t = 35$, $M = M_0 e^{-0\cdot0231\times35}$
$$= M_0 \times 0{\cdot}445525854\ldots$$

Therefore 45% of the original plant food is left after 35 days.

(d) When $M = \frac{1}{4}M_0$:

$$\tfrac{1}{4}M_0 = M_0 e^{-0\cdot0231t}$$
$$\Rightarrow e^{-0\cdot0231t} = \tfrac{1}{4}$$
$$-0{\cdot}0231t = \ln\left(\tfrac{1}{4}\right)$$
$$t = -\tfrac{1}{0\cdot0231}\ln\left(\tfrac{1}{4}\right)$$
$$= 60{\cdot}01274291\ldots$$
$$> 60$$

Therefore the manufacturer's claim is justified.

Question B1

$149 = 1 \times 139 + 10$

$139 = 13 \times 10 + 9$

$10 = 1 \times 9 + 1$

Now working backwards...

$$\begin{aligned}1 &= 10 - 1 \times 9 \\ &= 10 - (139 - 13(10)) \\ &= 14(10) - 139 \\ &= 14(149 - 139) - 139 \\ &= 14(149) - 15(139)\end{aligned}$$

Therefore $149x + 139y = 1$ with $x = 14$ and $y = -15$.

Question B2

$$\frac{dy}{dx} + \frac{y}{x} = x$$

$$\frac{dy}{dx} + \frac{1}{x}y = x$$

Integrating factor is $e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$

Therefore $x\frac{dy}{dx} + y = x^2$

so $\frac{d}{dx}(xy) = x^2$

and $xy = \int x^2 \, dx$

$= \frac{1}{3}x^3 + c$

So the general solutions is $y = \frac{1}{3}x^2 + \frac{c}{x}$.

Question B3

$$AB = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+4-3 & 0-2+2 & 1-2+1 \\ 1+8-9 & 0-4+6 & 1-4+3 \\ 1-4+3 & 0+2-2 & 1+2-1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= 2\mathbb{I}$$

(i)

$$\begin{aligned} AB &= 2\mathbb{I} \\ A^{-1}AB &= A^{-1}2\mathbb{I} \\ B &= 2A^{-1} \\ A^{-1} &= \tfrac{1}{2}B \\ &= \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 4 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix} \end{aligned}$$

(ii) $A^2B = A.AB = A.2\mathbb{I} = 2A$

So $A^2B = 2\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 6 \\ 2 & -2 & -2 \end{pmatrix}$

Question B4

Method 1

$$f(x) = (2+x)\ln(2+x) \quad \Rightarrow \quad f(0) = 2\ln 2$$
$$f'(x) = \ln(2+x) + 1 \quad \Rightarrow \quad f'(0) = \ln 2 + 1$$
$$f''(x) = \frac{1}{2+x} \quad \Rightarrow \quad f''(0) = \tfrac{1}{2}$$
$$f'''(x) = -\frac{1}{(2+x)^2} \quad \Rightarrow \quad f'''(0) = -\tfrac{1}{4}$$

Using the Maclaurin series expansion…

$$\begin{aligned} f(x) &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) \\ &= 2\ln 2 + x\ln 2 + x + \tfrac{1}{4}x^2 - \tfrac{1}{24}x^3 \\ &= 2\ln 2 + (1+\ln 2)x + \frac{x^2}{4} - \frac{x^3}{24} \end{aligned}$$

Method 2

$$f(x) = \ln(2+x) \quad \Rightarrow \quad f(0) = \ln 2$$
$$f'(x) = \frac{1}{2+x} \quad \Rightarrow \quad f'(0) = \tfrac{1}{2}$$
$$f''(x) = -\frac{1}{(2+x)^2} \quad \Rightarrow \quad f''(0) = -\tfrac{1}{4}$$
$$f'''(x) = \frac{2}{(2+x)^3} \quad \Rightarrow \quad f'''(0) = \tfrac{1}{4}$$

Using the Maclaurin series expansion…

$$\begin{aligned} f(x) &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) \\ &= \ln 2 + x + \tfrac{1}{4}x^2 - \tfrac{1}{24}x^3 \end{aligned}$$

So the Maclaurin series expansion for…

$$\begin{aligned} (2+x)\ln(2+x) &= (2+x)\left(\ln 2 + x + \tfrac{1}{4}x^2 - \tfrac{1}{24}x^3\right) \\ &= 2\ln 2 + x - \tfrac{1}{4}x^2 + \tfrac{1}{2}x^3 + x\ln 2 + \tfrac{1}{2}x^2 - \tfrac{1}{8}x^3 + \tfrac{1}{24}x^4 \\ &= 2\ln 2 + x + x\ln 2 + \tfrac{1}{4}x^2 - \tfrac{1}{24}x^3 \\ &= 2\ln 2 + (1+\ln 2)x + \tfrac{1}{4}x^2 - \tfrac{1}{24}x^3 \end{aligned}$$

Question B5

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-3y=6x-1$$

Auxiliary Equation...

$$m^2+2m-3=0$$
$$(m+3)(m-1)=0$$
$$m=-3 \quad \text{or} \quad m=1$$

Therefore the complimentary function is $y=c_1e^x+c_2e^{-3x}$

Particular Integral...

$$y=Ax+B$$
$$\frac{dy}{dx}=A$$
$$\frac{d^2y}{dx^2}=0$$

$$\therefore \quad 0+2A-3Ax-3B=6x-1$$

Equating coefficients...

$$-3A=6$$
$$A=2$$

$$2A-3B=-1$$
$$-4-3B=-1$$
$$3B=-3$$
$$B=-1$$

Therefore the Particular Integral is $y=-2x-1$ and the general solution is...

$$y=c_1e^x+c_2e^{-3x}-2x-1$$

Question B6

(a) (i) Given $L_1:\ x = 8 - 2t$

$$y = -4 + 2t$$
$$z = 3 + t$$

and from $L_2:\ \frac{x}{-2} = \frac{y+2}{-1} = \frac{z-9}{2} = s$ we get $x = -2s$

$$y = -s + 2$$
$$z = 2s + 9$$

At point of intersection…

$$8 - 2t = -2s \quad \Rightarrow \quad 2s - 2t = 8 \quad ①$$
$$-4 + 2t = -s - 2 \quad \Rightarrow \quad s + 2t = 2 \quad ②$$
$$3 + t = 2s + 9 \quad \Rightarrow \quad 2s - t = -6 \quad ③$$

$2\times②-①$: $3t = 6$

$$t = 2$$

Substitute $t = 2$ into equation ①…

$$s - 2 = -4$$
$$s = -2$$

Do $s = -2$ and $t = 2$ satisfy ③?

$$2s - t = 2(-2) - 2$$
$$= -6$$

When $t = 2$: $x = 8 - 4 = 4$

$$y = -4 + 4 = 0$$
$$z = 3 + 2 = 5$$

Therefore the lines L_1 and L_2 intersect at the point $(4, 0, 5)$.

(ii) From L_1: $x = 8 - 2t \quad \Rightarrow \quad t = \frac{x-8}{-2}$

$$y = -4 + 2t \quad \Rightarrow \quad t = \frac{y+4}{2}$$
$$z = 3 + t \quad \Rightarrow \quad t = \frac{z-3}{1}$$

Giving direction cosine $\boldsymbol{d}_1 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$, and from L_2 we get $\boldsymbol{d}_2 = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$.

Let an angle between lines L_1 and L_2 be θ.

$$\begin{aligned}\cos\theta &= \frac{\boldsymbol{d}_1.\boldsymbol{d}_2}{|\boldsymbol{d}_1||\boldsymbol{d}_2|}\\ &= \frac{(-2)(-2)+2(-1)+1\times 2}{\sqrt{(-2)^2+2^2+1^2}\sqrt{(-2)^2+(-1)^2+2^2}}\\ &= \frac{4-2+2}{\sqrt{9}\sqrt{9}}\\ &= \tfrac{4}{9}\end{aligned}$$

$$\therefore\ \theta = \cos^{-1}\left(\tfrac{4}{9}\right)$$

(b) (i) Since direction cosine of L_2 is $\begin{pmatrix}-2\\-1\\2\end{pmatrix}$, plane has equation of the form

$-2x-y+2z=d$.

Since $(1,-4,2)$ lies on the plane...

$$\begin{aligned}d &= -2\times 1-(-4)+2\times 2\\ &= -2+4+4\\ &= 6\end{aligned}$$

Thus the equation of the plane is...

$$-2x-y+2z=6 \text{ or } 2x+y-2z+6=0$$

(ii) Substituting parametric form of line L_1 into plane Π...

$$\begin{aligned}-2x-y+2z&=6\\ -2(8-2t)-(-4+2t)+2(3+t)&=6\\ -16+4t+4-2t+6+2t&=6\\ 4t-6&=6\\ t&=3\end{aligned}$$

When $t=3$, $x=8-2\times 3=2$

$y=-4+2\times 3=2$

$z=3+3=6$

Therefore the point of intersection is $(2,2,6)$.