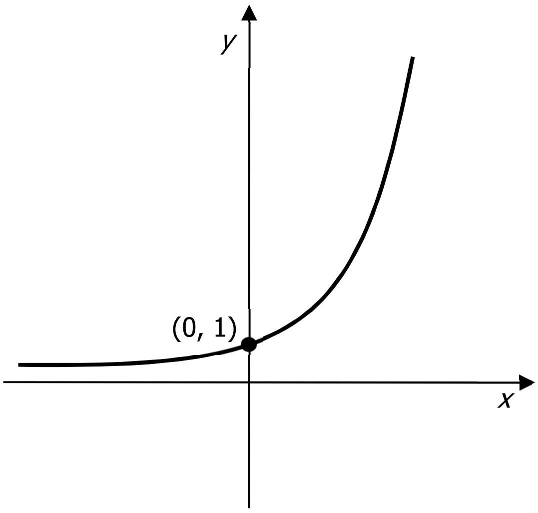
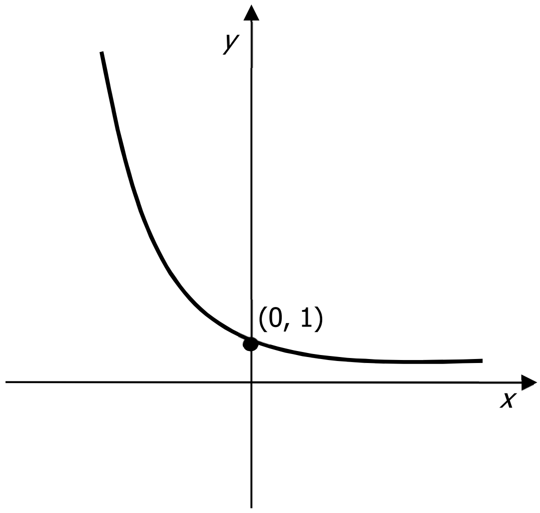
Exponential and Logarithmic Functions

Exponential functions are those with variable powers, e.g. $y = a^x$. Their graphs take two forms:





When a > 1, the graph:

- is always increasing
- is always positive
- never cuts the *x* – axis
- passes through (0, 1)
- shows **exponential growth**

When 0 < a < 1, the graph

- is always decreasing
- is always positive
- never cuts the *x* – axis
- passes through (0, 1)
- shows **exponential decay**

Exponential Functions as Models

Example 1: After distilling, whisky is aged in wooden barrels for a number of years before bottling. During aging, 2% of the volume is lost each year due to evaporation (known as "the Angels' Share").

What percentage volume is lost to evaporation in a barrel of whisky aged for 12 years?

Example 2: In computing, the number of transistors on a microchip is an indication of the power of the computer. Moore's Law predicted that the number of transistors on commercially available microchips will double every **two** years.

The Intel 4004 was released in 1971, and contained 2003 transistors.

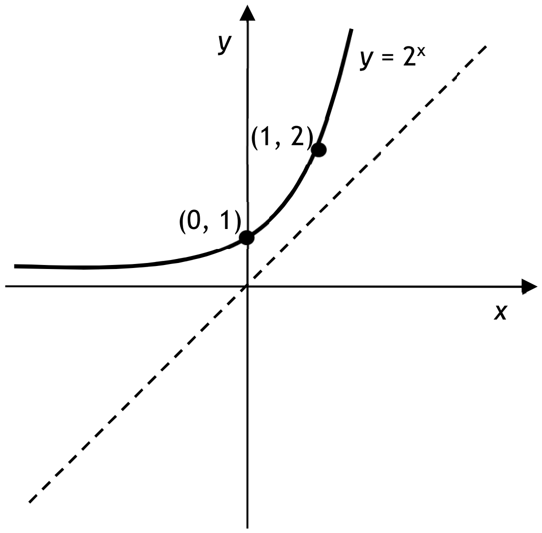
a) Write down a formula which describes the number of transistors on a microchip released *n* years after the Intel 4004.

b) Calculate:

(i) The number of transistors on the Intel 486 (released 1989)

(ii) the increase in power between the Intel 4004 and the Intel Core i3 (released 2007).

Logarithmic Functions



The inverse of an exponential function is known as a **logarithmic** function.

If $f(x) = a^x$, then $f^{-1}(x) = \log_a x$

("log to the base a of x")

We have seen that the graph of the inverse of a function can be obtained by reflection in the line $y = x$.

Since the graph of $y = 2^x$ passes through the points (0, 1) and (1, 2), then the inverse of $f(x) = 2^x$ must pass through the points (1, 0) and (2, 1).

Example 3 Add the graph of $y = \log_2 x$ to the graph opposite.

Note that:

$y = a^x$ passes through (0, 1) and (1, a)

$y = \log_a x$ passes through (1, 0) and (a, 1)

$y = a^x$ means "a multiplied by itself x times gives y "

$y = \log_a x$ means "y is the number of times I multiply a by itself to get x "

Since the graph does not cross the y-axis, we can only take the logarithm of a positive number

The expression "$\log_a x$ "can be read as "a to the power of what is equal to x ?", e.g. $\log_2 8$ means "2 to the power of what equals 8?", so $\log_2 8 = 3$.

Example 4: Write in logarithmic form:

a) $5^2 = 25$

b) $12^1 = 12$

c) $8^{1/3} = 2$

d) $8^x = y$

e) $1 = q^0$

f) $(x - 3)^4 = k$

Example 5: Write in exponential form:

a) $3 = \log_5 125$

b) $\log_3 81 = 4$

c) $\log_4 4096 = 6$

d) $\log_2\left(\frac{1}{4}\right) = -2$

e) $\log_b g = 5h$

f) $1 = \log_7 7$

Example 6: Evaluate:

a) $\log_8 64$

b) $\log_2 32$

c) $\log_{3.5} 3.5$

d) $\log_{25} 5$

e) $\log_4\left(\frac{1}{2}\right)$

Since $a^1 = a$, then **$\log_a a = 1$**

Since $a^0 = 1$, then **$\log_a 1 = 0$.**

Laws of Logarithms

$\log_a xy = \log_a x + \log_a y$	$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$	$\log_a x^n = n \log_a x$

Example 7: Simplify:

a) $\log_2 4 + \log_2 8 - \log_2 \frac{1}{2}$

b) $2\log_5 10 - \log_5 4$

c) $\frac{1}{4}(\log_3 810 - \log_3 10)$

Solving Logarithmic Equations

You **MUST** memorise the laws of logarithms to solve log equations! As we can only take logs of **positive** numbers, we must remember to discard any answers which violate this rule!

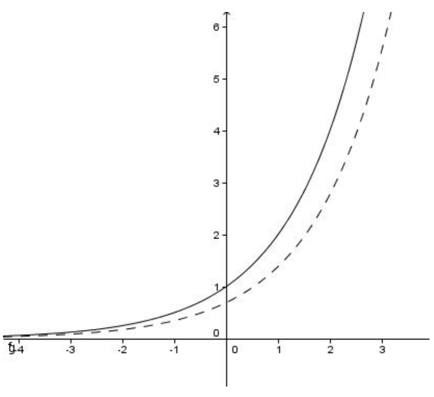
Example 8: Solve:

a) $\log_4 (3x - 2) - \log_4 (x + 1) = \frac{1}{2}$ $\left(x > \frac{2}{3}\right)$

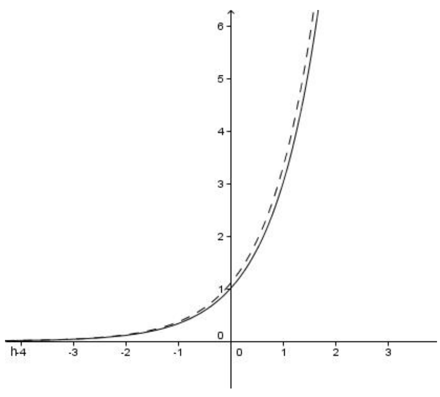
b) $\log_6 x + \log_6 (2x - 1) = 2$ $\left(x > \frac{1}{2}\right)$

The Exponential Function and Natural Logarithms

The graph of the derived function of $y = a^x$ can be plotted and compared with the original function. The new graphs are also exponential functions. Below are the graphs of $y = 2^x$ and $y = 3^x$ (solid lines) and their derived functions (dotted).



$f(x) = 2^x$



$f(x) = 3^x$

The derived function of $y = 2^x$ lies **under** the original graph, but the derived function of $y = 3^x$ lies **above** it.

This means that there must be a value of a between 2 and 3 where the derived function lies **on** the original.

i.e. where $f(x) = f'(x)$

The value of the base of this function is known as e, and is approximately 2.71828.

The function $y = e^x$ is known as The Exponential Function.

The function $y = \log_e x$ is known as the Natural Logarithm of x, and is also written as $\ln x$.

Example 9: Evaluate:

a) e^3

b) $\log_e 120$

Example 10: Solve:

a) $\ln x = 5$

b) $5^{x-1} = 16$

Example 11: Atmospheric pressure at various heights above sea level can be determined by using the formula

$$P_t = P_0 e^{-0.035t}$$

where:
P_0 is the pressure at sea level (defined as 1 standard atmosphere)
P_t is the pressure at height t
t is the height above sea level in thousands of feet

a) Find the air pressure in standard atmospheres at a height of 7500 feet above sea level.

b) Find the height at which P_t is 10% of that at sea level.

Example 12: A radioactive element decays according to the law $A_t = A_0 e^{kt}$, where A_t is the number of radioactive nuclei present at time t years and A_0 is the initial amount of radioactive nuclei.

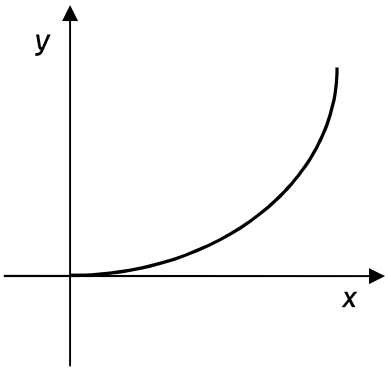
a) After 150 years, 240g of this material had decayed to 200g. Find the value of k accurate to 3 s.f.

b) The half-life of the element is the time taken half the mass to decay. Find the half-life of the material.

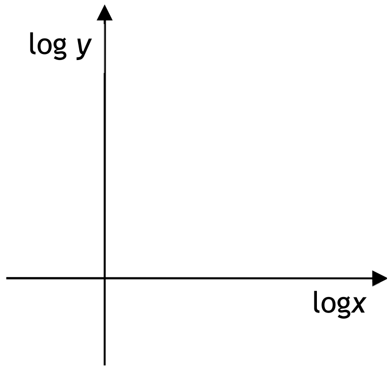
Using Logs to Analyse Data, Type 1: $y = kx^n \Leftrightarrow \log y = n \log x + \log k$

When the data obtained from an experiment results in an exponential graph of the form $y = kx^n$ as shown below, we can use the laws of logarithms to find the values of k and n.

To begin, take logs of both sides of the exponential equation.

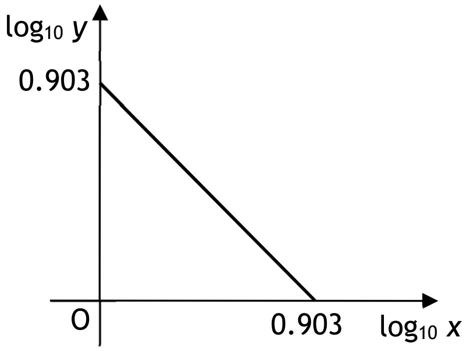


$y = kx^n$



$y = kx^n$ **This gives a straight line graph!** $\log y = n \log x + \log k$

Note: the choice of base is not important, as long as the **same** base is used on both sides.



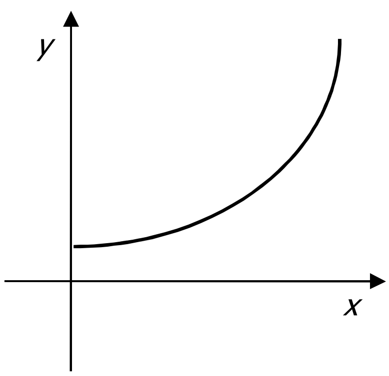
Example 13: Data are recorded from an experiment and the graph opposite is produced.

a) Find the equation of the line in terms of $\log_{10} x$ and $\log_{10} y$.

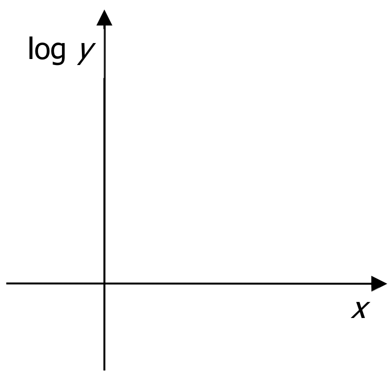
b) Hence express y in terms of x .

Using Logs to Analyse Data, Type 2: $y = kn^x \Leftrightarrow \log y = \log n\ (x) + \log k$

A similar technique can be used when the graph is of the form $y = kn^x$ (i.e. x is the index, not the base as before).

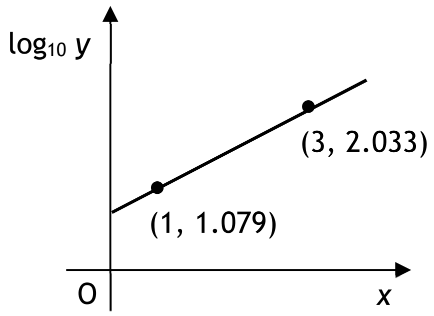


$y = kn^x$



$y = kn^x$ $\log y = (\log n)\, x + \log k$

Example 14: The data below are plotted and the graph shown is obtained.



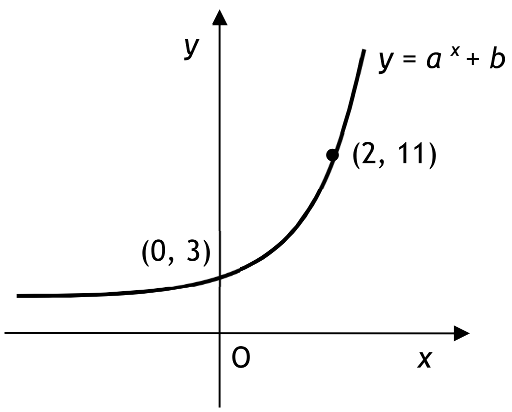
a) Find the equation of the line in terms of $\log_{10} y$ and x.

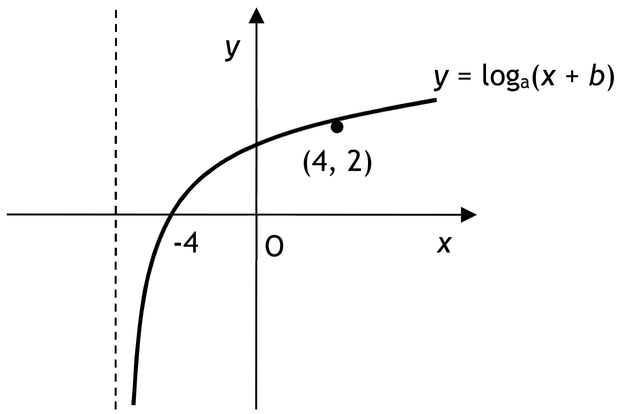
b) Hence express y in terms of x .

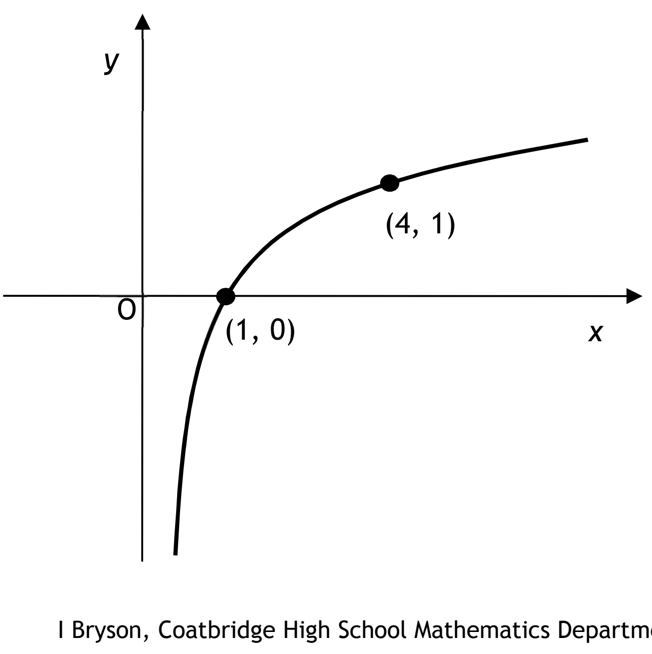
Related Graphs of Exponentials and Logs

See the summary on page 16 on transformation of graphs.

Example 15: Determine the values of a and b in each graph below.







Example 16: The graph of $y = \log_4 x$ is shown. On the same diagram, sketch:

a) $y = \log_4 4x$

b) $y = \log_4 \left(\frac{1}{4x}\right)$

Past Paper Example 1:

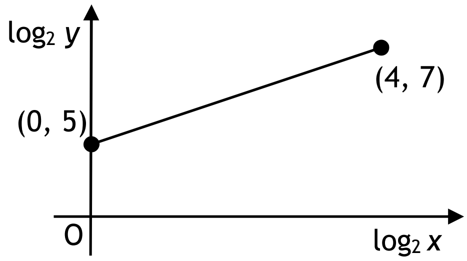
a) Show that $x = 1$ is a root of $x^3 + 8x^2 + 11x - 20 = 0$, and hence factorise $x^3 + 8x^2 + 11x - 20$ fully

b) Solve $\log_2(x + 3) + \log_2(x^2 + 5x - 4) = 3$

Past Paper Example 2: Variables x and y are related by the equation $y = kx^n$.

The graph of $\log_2 y$ against $\log_2 x$ is a straight line through the points (0, 5) and (4, 7), as shown in the diagram.

Find the values of k and n.



Past Paper Example 3: The concentration of the pesticide *Xpesto* in soil is modelled by the equation:

$$P_t = P_0 e^{-kt}$$

where: P_0 is the initial concentration
P_t is the concentration at time t
t is the time, in days, after the application of the pesticide.

a) Once in the soil, the half-life of a pesticide is the time taken for its concentration to be reduced to one half of its initial value.

If the half-life of *Xpesto* is 25 days, find the value of k to 2 significant figures.

b) Eighty days after the initial application, what is the percentage decrease in *Xpesto*?