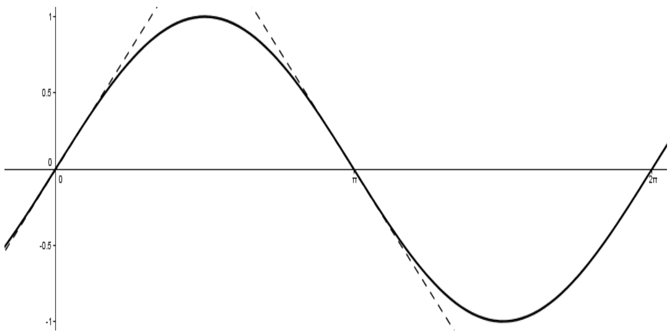
Further Calculus

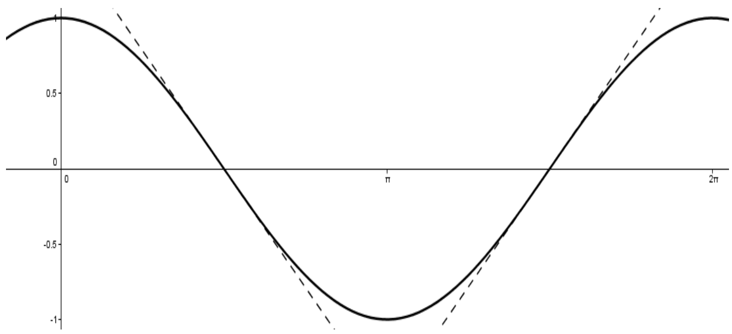
Let $f(x)$ = sin$x$ and $g(x)$ = cos$x$. The graphs of $y = f(x)$ and $y = g(x)$ are shown below, where the x-axis is measured in radians. Tangents to each curve have been drawn at the following points:

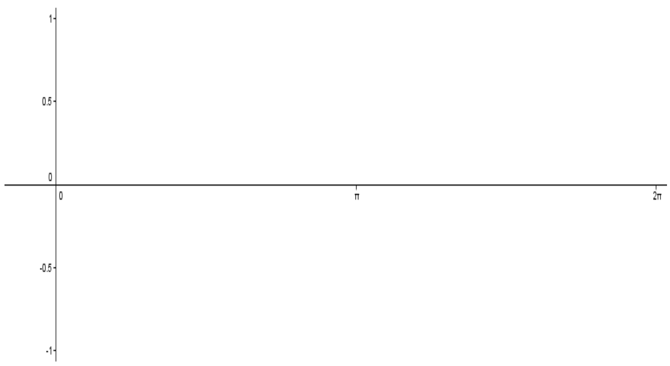
On y = sin$x$, the tangent at $x = 0$ has $m = 1$, and the tangent at $x = \pi$ has $m = -1$.

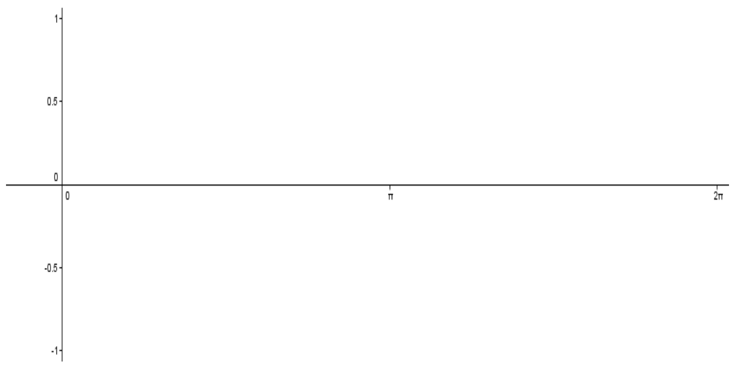
On y = cos$x$, the tangent at $x = \dfrac{\pi}{2}$ has $m = -1$, and the tangent at $x = \dfrac{3\pi}{2}$ has $m = 1$.

Draw the graphs of $y = f'(x)$ and $y = g'(x)$ below.









The graphs of the derived functions therefore show that:

If y = sinx , $\dfrac{dy}{dx}$ =

If y = cosx , $\dfrac{dy}{dx}$ =

Example 1: Find the derivative in each case:

a) $y = 4\sin x$

b) $f(x) = 2\cos x$

c) $g(x) = -\dfrac{1}{2}\cos x$

d) $h = -5\sin k$

As integration is the opposite of differentiation, we can also say that:

$\int \cos x \, dx =$

$\int \sin x \, dx =$

Example 2: Find:

a) $\int 24\cos x \, dx$

b) $\int -3\sin s \, ds$

c) $\int (3x - \cos x) dx$

IMPORTANT!

- **Definite Integrals of sin and cos functions <u>MUST</u> be done in radians!**
- **NEVER ignore any brackets where the limit is zero!**

Example 3: Evaluate:

a) $\int_0^{\pi/2} \sin x \, dx$

b) $\int_0^{\pi/4} (\sin x - \cos x) \, dx$

The Chain Rule

By first expanding the brackets, find the derivatives of the following functions:

a) $y = (3x + 1)^2$

$\therefore \ \frac{dy}{dx} =$ ___$(3x + 1)$ X _____

b) $y = (2x^2 - 1)^2$

$\therefore \ \frac{dy}{dx} =$ ___$(2x^2 - 1)$ X ______

c) $y = (2x + 1)^3$

$\therefore \ \frac{dy}{dx} =$ ___$(2x + 1)^2$ X ______

In each case, we can factorise the answer to give us back the original function, which has been differentiated as if it was just an x^2 or x^3 term (multiply by the old power, drop the power by one), and then **multiplied by the derivative of the function in the bracket.**

This is known as the **Chain Rule,** and can be written generally for brackets with powers as:

For $f(x) = a(\ldots\ldots\ldots)^n$, $f'(x) = an(\ldots\ldots\ldots)^{n-1}$ x (DOB)

where DOB = the Derivative Of the Bracket

Example 4: Use the chain rule to differentiate:

a) $f(x) = (4x - 2)^4$

b) $y = \sin^2 x$

c) $g(x) = \dfrac{1}{\sqrt{2x^2 + x}}$ $\left(x < -\frac{1}{2},\ x > 0\right)$

The Chain Rule can also be applied to sine and cosine functions with double or compound angles, or to more complicated composite functions containing sine and cosine.

For functions including sine and cosine components:

For f(x) = sin(……), f'(x) = cos(……) x DOB	For f(x) = cos(……), f'(x) = - sin(……) x DOB

Example 5: Differentiate:

a) $y = \sin(3x)$

b) $f(x) = \cos\left(\frac{\pi}{4} - 2x\right)$

c) $y = \sin(x^2)$

Example 6: Find the equation of the tangent to $y = \sin\left(2x + \frac{\pi}{3}\right)$ when $x = \frac{\pi}{6}$.

Further Integration

We have seen that integration is **anti-differentiation**, i.e. the opposite of differentiating.

As finding the derivative of a function with a bracket included multiplying by DOB, then integrating must also include **dividing** by DOB.

To integrate:

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)\times a} + C$$

Important Point: Integration is more complicated than differentiation!

This method only works for **linear** functions inside the bracket, i.e. the highest power = 1. To find, e.g., $\int \left(g^3 - 7\right)^2 dg$, we would have to multiply out the bracket and integrate each term separately.

Example 7: Evaluate:

a) $\int (x+3)^3 dx$

b) $\int (4x - 7)^9 dx$

c) $\int \frac{dt}{(4t+9)^2} \quad \left(t \neq -\frac{9}{4}\right)$

d) $\int_1^2 (2t+5)^3\, dt$

e) $\int_0^6 \frac{dx}{\sqrt{4x+1}} \quad \left(x > -\frac{1}{4}\right)$

For functions including sine and cosine components:

$$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)+C$$

$$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+C$$

Example 8: Evaluate:

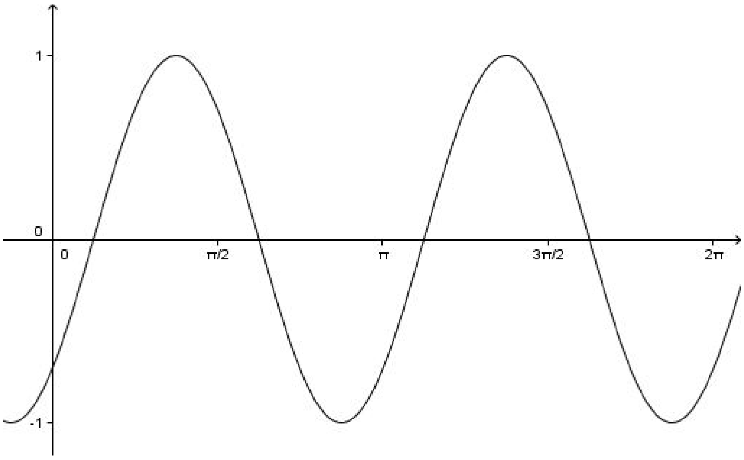
a) $\int \sin 4x\, dx$

b) $\int 3\cos 2x\, dx$

c) $\int \sin(1-2x)\, dx$

d) Evaluate $\int_0^{2\pi} \sin\left(\frac{1}{2}x\right) dx$

Example 9: Find the area enclosed by $y = \sin\left(2x - \frac{\pi}{4}\right)$, the x – axis and the lines $x = 0$ and $x = \frac{\pi}{2}$.



Differentiation in Context: Optimisation

Differentiation can be used to find the maximum or minimum values of things which happen in real life. Finding the maximum or minimum value of a system is called **optimisation.**

Example 10: A carton is in the shape of a cuboid with a rectangular base and a volume of 3888cm^3.

The surface area of the carton can be represented by the formula $A(x) = 4x^2 + \dfrac{5832}{x}$.

Find the value of x such that the surface area is a minimum.

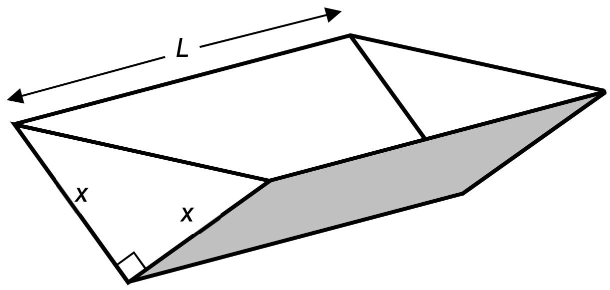
In exams, optimisation questions almost always consist of two parts: part one asks you to **show** that a situation can be described using an algebraic formula or equation, whilst part two asks you to use the given formula to find a maximum or minimum value by differentiation.

Leave part 1 of an optimisation question until the end of the exam (if you have time), as they are almost always (i) more difficult than finding the stationary point and (ii) worth fewer marks.

Remember that part 2 is just a well-disguised "find the minimum/maximum turning point of this function" question!

Example 11: An open water tank, in the shape of a triangular prism, has a capacity of 108 litres. The tank is to be lined on the inside in order to make it watertight.

The triangular cross-section of the tank is right-angled and isosceles, with equal sides of length x cm. The tank has a length of L cm.



a) Show that the surface area to be lined, A cm^2, is given by $A(x) = x^2 + \dfrac{432000}{x}$

b) Find the minimum surface area of the tank.

Uses of Calculus in Real Life Situations

In the same way that geometry is the study of shape, calculus is the study of how functions change. This means that wherever a system can be described mathematically using a function, calculus can be used to find the ideal conditions (as we have seen using optimisation) or to use the rate of change at a given time to find the total change (using integration).

As a result, calculus is used throughout the sciences: in Physics (Newton's Laws of Motion, Einstein's Theory of Relativity), Chemistry (reaction rates, radioactive decay), Biology (modelling changes in population), Medicine (using the decay of drugs in the bloodstream to determine dosages), Economics (finding the maximum profit), Engineering (maximising the strength of a building whilst using the minimum of material, working out the curved path of a rocket in space) and more.

Example 12: In Physics, the formulae for kinetic energy (E_k) and momentum (p) are

$$E_k = \frac{1}{2}mv^2 \qquad \text{and} \qquad p = mv$$

respectively.

a) How could the formula for momentum be obtained from the formula for kinetic energy?

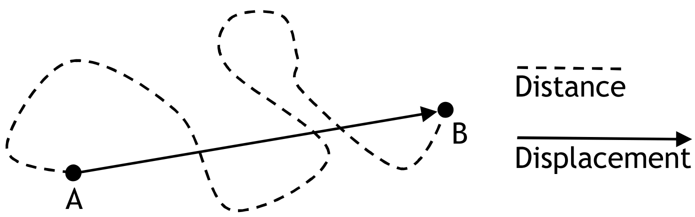
b) How could the formula for kinetic energy be obtained from the formula for momentum?

Displacement, Velocity and Acceleration

The most common use of this approach considers the link between displacement, velocity and acceleration.

When an object moves on a journey, we normally think of the total distance travelled.

Displacement is the straight line distance between the start and end points of a journey (so the displacement is not necessarily the same as the distance travelled!)



As displacement is a "straight-line" measurement, it involves direction and therefore is a **vector** quantity: another name for displacement is the **position**.

Velocity is the vector equivalent of speed, i.e. if speed is a measure of the distance travelled in a given time, then velocity is a measure of the change in displacement which occurs in a given time.

Velocity is defined as the **rate of change of displacement with respect to time.**

Acceleration measures the change in velocity of an object in a given time: if two race cars have the same top speed, then the one which can get to that top speed first would win a race.

Acceleration is defined as the **rate of change of velocity with respect to time.**

If one of either displacement, velocity or acceleration can be described using a function, then the other two can be obtained using either differentiation or integration, i.e.:

$$\begin{tikzcd}
\text{Displacement} \arrow[r, bend left, "\text{Differentiate}"] & \text{Velocity} \arrow[l, bend left, "\text{Integrate}"] \arrow[r, bend left, "\text{Differentiate}"] & \text{Acceleration} \arrow[l, bend left, "\text{Integrate}"]
\end{tikzcd}$$

Example 13: The displacement s cm at a time t seconds of a particle moving in a straight line is given by the formula $s = t^3 - 2t^2 + 3t$.

a) Find its velocity v cm/s after 3 seconds.

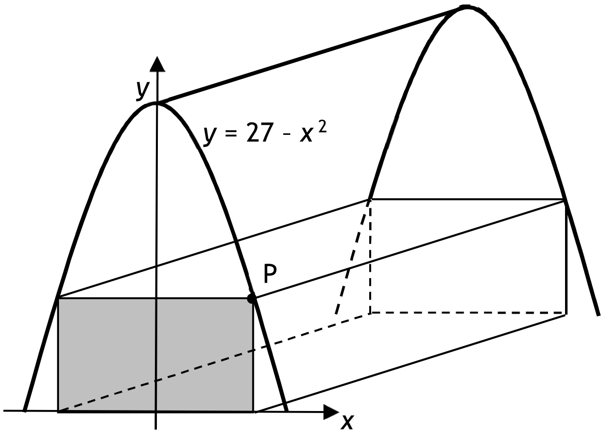
b) The time at which its acceleration a is equal to 26cm/s^2.

Example 14: The velocity of an electron is given by the formula $v(t) = 5 \sin \left(2t - \frac{\pi}{4}\right)$.

a) Find the first time that the velocity of the elctron is at its maximum.

b) Find a formula for the displacement of the electron, given that $s = 0$ when $t = 0$.

Past Paper Example 1: An architect has designed a new open-plan office building using two identical parabolic support beams spaced 25m apart as shown below. The front beam, relative to suitable axes, has the equation $y = 27 - x^2$. The inhabited part of the building is to take the shape of a cuboid.



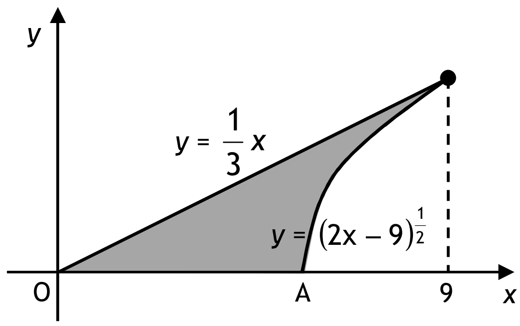
a) By considering the point P in the corner of the front face of the building, show that the area of this face is given by $\boldsymbol{A(x) = 54x - 2x^3}$.

b) Find the maximum volume of the inhabited section of the building.

Past Paper Example 2: A curve has equation $y = (2x - 9)^{\frac{1}{2}}$

Part of the curve is shown in the diagram opposite.

a) Show that the tangent to the curve at the point where $x = 9$ has equation $y = \dfrac{1}{3}x$.



b) Find the coordinates of A, and hence find the shaded area.

Past Paper Example 3:

a) By writing $\sin 3x$ as $\sin (2x + x)$, show that

$$\sin 3x = 3 \sin x - 4 \sin^3 x.$$

b) Hence find $\int \sin^3 x \, dx$