Polynomials

A **polynomial** is an expression with terms of the form ax^n, where n is a whole number. For example, $5p^4 - 3p^3$ is a polynomial, but $3p^{-1}$ or $\sqrt[3]{p^2}$ are not.

The **degree** (or order) of a polynomial is its highest power, e.g. the example above has a degree of 4.

The number part of each term is called its **coefficient**, e.g. the coefficients of p^4, p^3 and p above are 5, -3 and 0 (as there is no p term!) respectively (note that $5p^4$ would also be a polynomial on its own, with coefficients of zero for all other powers of p).

Evaluating Polynomials

An easy way to find out the value of a polynomial function is by using a **nested table.**

Example 1: Evaluate f(4) for $f(x) = 2x^4 - 3x^3 - 10x^2 - 5x + 7$.

4	2	-3	-10	-5	7
	↓				
	2				

← Line up coefficients

Example 2: Evaluate f(-1) for $f(x) = 3x^5 - 2x^3 + 4$.

-1	3	0	-2	0	0	4
	↓					
	3					

Missing powers have coefficients of zero!

Synthetic Division

Dividing 67 by 9 gives an answer of "7 remainder 4". We can write this in two ways:

67 ÷ 9 = 7 remainder 4 **OR** **9 x 7 + 4 = 67**

For this problem, 9 is the **divisor**, 7 is the **quotient**, and 4 is the **remainder** (note that if we were dividing 63 by 9, the remainder would be zero, since 9 is a **factor** of 63).

Example 3:

a) Remove brackets and simplify:

$(x - 3)(x^2 + 9x - 12) + 11$

b) Evaluate f(3) for $f(x) = x^3 + 6x^2 - 39x + 47$

This shows that: $x^3 + 6x^2 - 39x + 47 = (x - 3)(x^2 + 9x - 12) + 11$

OR

$(x^3 + 6x^2 - 39x + 47) \div (x - 3) = (x^2 + 9x - 12)$ remainder 11

Compare the numbers on the **bottom row** of the nested table in part b) with the coefficients in part a). This shows that we can use nested tables to divide polynomial expressions to give both the quotient and remainder (if one exists). This process is known as **synthetic division.**

Example 4: Find the remainder on dividing $x^3 - x^2 - x + 5$ by $(x + 5)$.

Example 5: Write $4p^4 + 2p^3 - 6p^2 + 3 \div (2p - 1)$ in the form $(ap - b)\,Q(p) + R$

Remainder Theorem and Factor Theorem

Considered together, these two theorems allow us to factorise algebraic functions (remember that a factor is a number or term which divides **exactly** into another, leaving **no remainder**).

If polynomial f(x) is divided by (x - h), then the remainder is f(h)

On division of polynomial f(x) by (x - h), if f(h) = 0, then (x - h) is a factor of f(x)

In other words, if the result of synthetic division on a polynomial by h is zero, then h is a **root** of the polynomial, and $(x - h)$ is a **factor** of it.

Example 6: $f(x) = 2x^3 - 9x^2 + x + 12$.

a) Show that $(x - 4)$ if a factor of f(x).

b) Hence factorise f(x) fully.

Example 7: Factorise fully $3x^3 + 2x^2 - 12x - 8$.

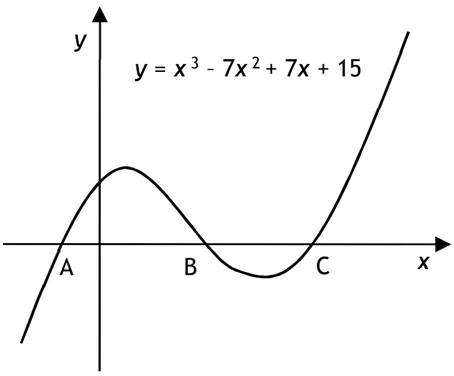
Example 8: Find the value of k for which $(x + 3)$ is a factor of $x^3 - 3x^2 + kx + 6$

Example 9: Find the values of a and b if $(x - 3)$ and $(x + 5)$ are both factors of $x^3 + ax^2 + bx - 15$

Solving Polynomial Equations

Polynomial equations are solved in exactly the same way as we solve quadratic equations: **make the right hand side equal to zero, factorise, and solve to find the roots.**

Example 10: The graph of the function $y = x^3 - 7x^2 + 7x + 15$ is shown. Find the coordinates of points A, B and C.

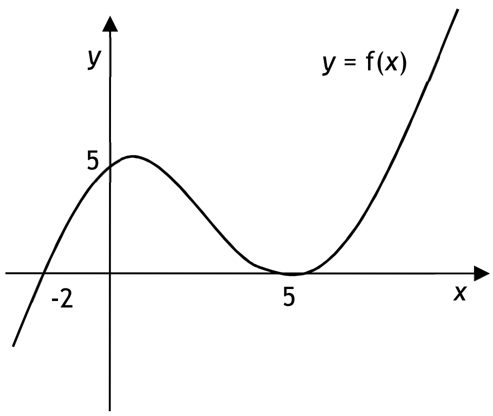


Finding a Function from its Graph: y = k(x – a)(x – b)(x – c)

This uses exactly the same system as that for quadratic graphs, but with more brackets (see page 33).

Remember: tangents to the *x* – axis have repeated roots!

Example 11: Find an expression for cubic function $f(x)$.



In exam-style questions, synthetic division is often used alongside other topics.

Example 12: A function is defined on the set of Real numbers by $f(x) = x^3 + 3x^2 - 9x + 5$

a) (i) Show that $x = 1$ is a root of $y = f(x)$.

(ii) Hence find the points of intersection between $y = f(x)$ and the coordinate axes.

b) Find the coordinates of the stationary points of $y = f(x)$ and determine their nature.

c) Sketch and annotate the curve $y = f(x)$.