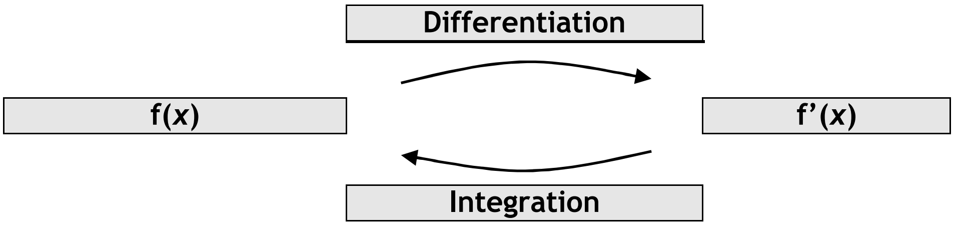
Integration

The reverse process to differentiation is known as **integration.**

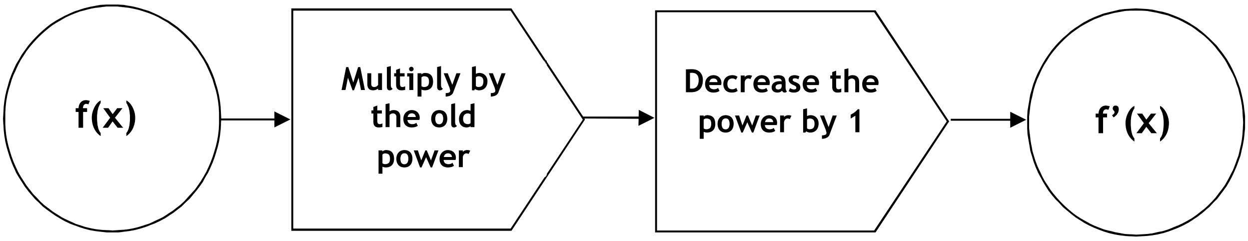


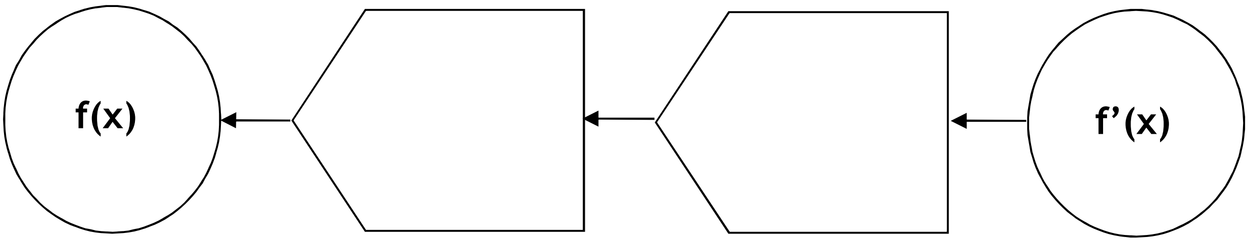
As it is the opposite of finding the derivative, the function obtained by integration is sometimes called the **anti-derivative,** but is more commonly known as the **integral**, and is given the sign $\int$.

If f(*x*) = x^n, then $\int x^n dx$ is "the integral of x^n with respect to *x* "

Indefinite Integrals and the Constant of Integration

To find the integral of a function, we do the **opposite** of what we would do to find the derivative:





Consider the three functions a(*x*) = $3x^2 + 2x + 5$, b(*x*) = $3x^2 + 2x - 8$ and c(*x*) = $3x^2 + 2x - \frac{13}{4}$.

In each case, the derivative of the function is the same, i.e. $6x + 2$. This means that $\int (6x+2)dx$ has more than one answer. Because there is more than one answer, we say that this is an **indefinite integral**, and we must include in the answer a constant value C, to represent the 5, -8, $-\frac{13}{4}$ etc which we would need to distinguish a(*x*) from b(*x*) from c (*x*) etc.

In general:

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C \qquad (n \neq -1)$$

IN **tegration** ***IN*** **creases the power!**

1. **Write as ax^n**
2. **Increase the power by 1**
3. **Divide by the new power**

Example 1: Find **(remember "+C"):**

a) $\int 2x\, dx$

b) $\int 4t^2\, dt$

c) $\int (3x^5 - 4)dx$

d) $\int \frac{3}{g^4}\,dg$ $(g \neq 0)$

e) $\int 6\sqrt[5]{p^3}\,dp$

f) $\int \frac{4y-3}{y^{2/3}}\,dy$ $(y \neq 0)$

The Definite Integral

A **definite integral** of a function is the difference between the integrals of f(x) at two values of x . Suppose we integrate f(x) and get F(x). Then the integral of f(x) when x = a would be F(a), and the integral when x = b would be F(b).

The definite integral of f(x), with respect to x , between a and b, is written as:

$$\int_a^b f(x)dx = F(b) - F(a) \qquad \text{(where b > a)}$$

For example, the integral of f(x) = $2x^2 - 4$ between the values x = -3 and x = 5 is written as $\int_{-3}^{5}(2x^2-4)\,dx$ and reads "the integral from -3 to 5 of $2x^2 - 4$ with respect to x ".

Note: definite integrals do NOT include the constant of integration!

$$\int_a^b f(x) = [F(b)+C] - [F(a)+C] = F(b) - F(a)$$

Example 2: Evaluate $\int_{-1}^{3}(2x-1)\,dx$

To find a definite integral:

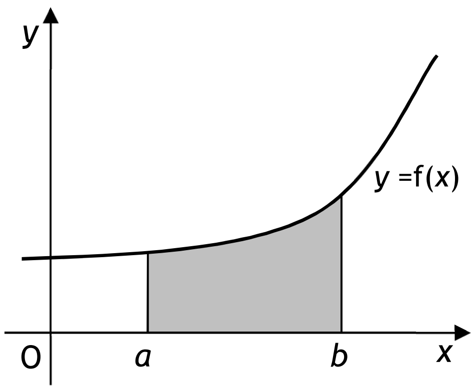
- **prepare the function for integration**
- **integrate as normal, but write inside square brackets with the limits to the right**
- **sub each limit into the integral, and subtract the integral with the lower limit from the one with the higher limit**

Example 3: Evaluate $\int_0^2 (p+1)(p-1)\,dp$

Example 4: Evaluate $\int_1^{\sqrt{3}} (x^2 - 2x)\,dx$

Example 5: Find the value of g such that $\int_{-2}^{g}(6x+5)\,dx = 6$.

Area Between a Curve and the *x* - axis.



In the diagram opposite, the area of the shaded section can be obtained by finding the area under the graph from O to b, and subtracting the area from O to a.

The value of each of these areas can be determined by integrating the function and substituting b or a respectively.

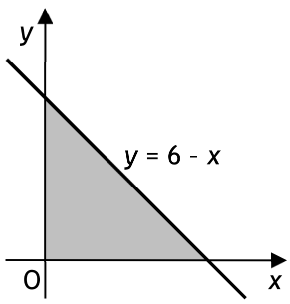
The area enclosed by the curve $y = f(x)$, the lines $y = a$, $y = b$ and the x - axis is equal to the definite integral of $f(x)$ between a and b

i.e. Area = $\int_{a}^{b} f(x)\,dx$

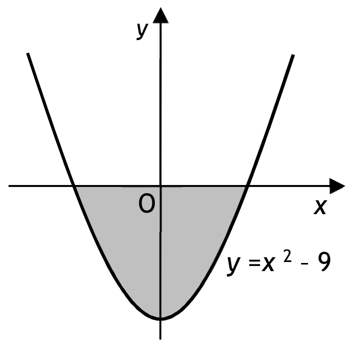
DON'T FORGET TO INCLUDE "dx" WHEN WRITING DOWN AN INTEGRAL!

Example 6: For each graph below,
(i) write down the integrals which describe the shaded regions
(ii) calculate the area of the shaded region

a)



b)

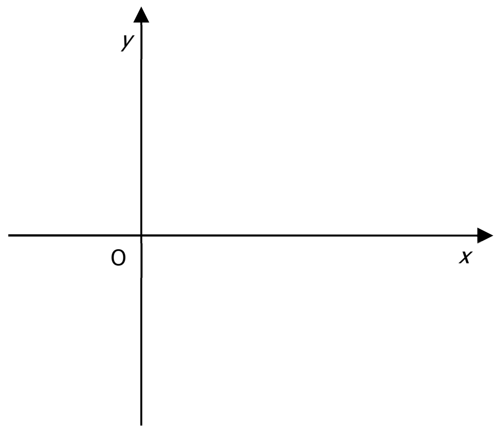


NOTE: Example 6b shows that areas UNDER the *x* - axis give NEGATIVE values!

Example 7:

a) Evaluate $\int_{-1}^{7}(2x-6)\,dx$

b) (i) Sketch below the area described by the integral $\int_{-1}^{7}(2x-6)\,dx$.

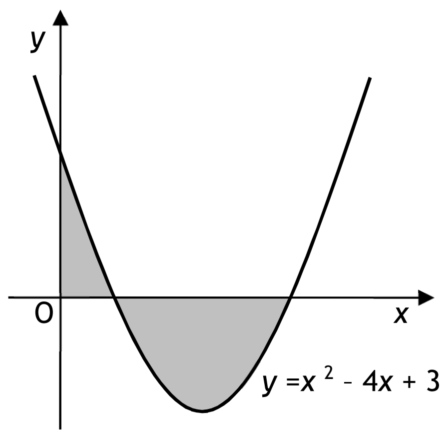


The answers for 7a and 7b do not match! This is because the area below the axis and the area above cancel each other out (as in 6b, definite integrals for areas below the *x* – axis produce negative values).

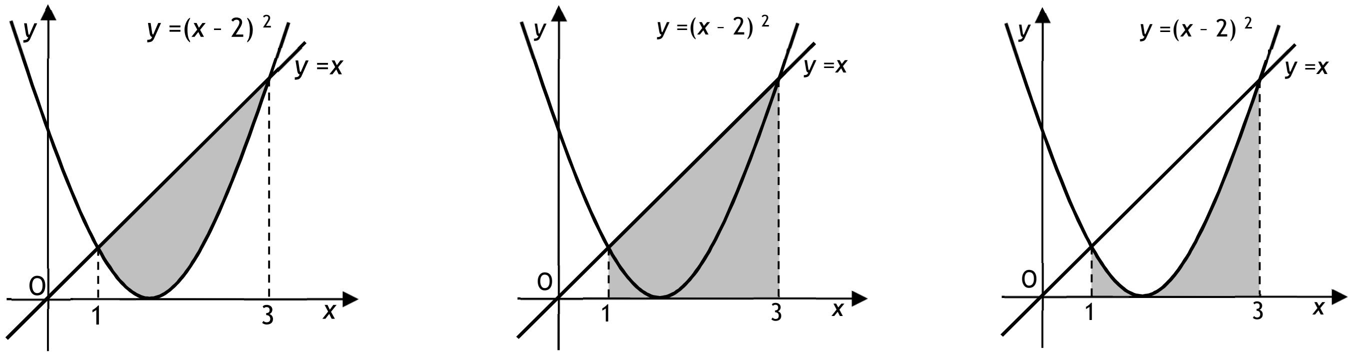
To find the area between a curve and the x-axis:

1. Determine the limits which describe the sections above and below the axis
2. Calculate areas separately
3. Find the total, *IGNORING THE NEGATIVE VALUE OF THE SECTION BELOW THE AXIS*.

Example 8: Determine the area of the regions bounded by the curve $y = x^2 - 4x + 3$ and the *x* – and *y* – axes.

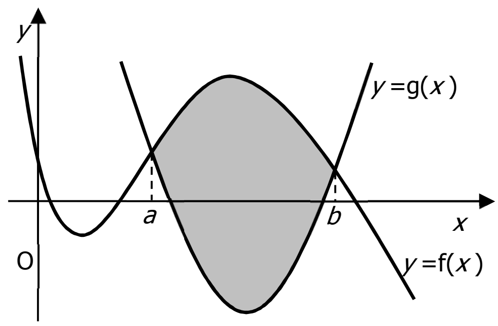


Consider the area bounded by the curves $y = (x - 2)^2$ and $y = x$.



$$\textbf{Area} = \int_1^3 x\,dx - \int_1^3 (x-2)^2\,dx$$

The diagrams above show that the area between the curves is equal to the area between the top function (x) and the x – axis MINUS the area between the bottom curve ($(x - 2)^2$) and the x – axis.



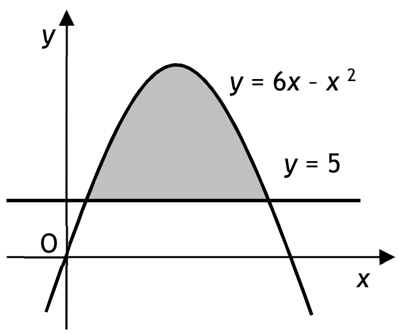
The area between the curves $y = f(x)$ and $y = g(x)$ (which meet at the points where $x = a$ and $x = b$) is given by:

$$A = \int_a^b (f(x) - g(x))\,dx$$

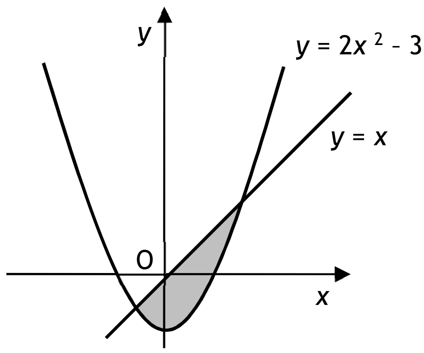
where:
- $f(x)$ is the TOP function and $g(x)$ is the BOTTOM
- $b > a$

Example 9: Write down the integrals used to determine the areas shown below:

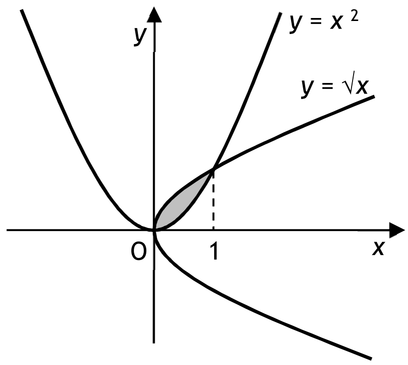
a)



b)



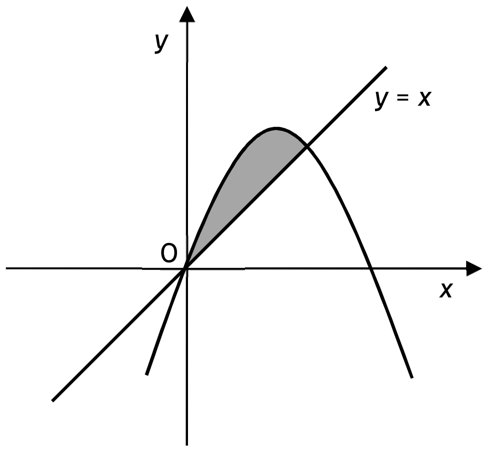
c)



To find the area between two curves:

1. **Make a sketch (if one has not been given)**
2. **Find points of intersection (make $y = y$ and solve)**
3. **Subtract the bottom function from the top function,** ***PUTTING THE BOTTOM FUNCTION IN BRACKETS!***
4. **Integrate**

Example 10: Find the area enclosed between the curve $y = 5x - x^2$ and the line $y = x$



Differential Equations

If we know the derivative of a function (e.g. $f'(x) = 6x^2 - 3$), we can obtain a formula for the original function in terms of x and C by integration. If we know the coordinates of a point on the **original** function, we can substitute into the integral to find the value of C and therefore the function in terms only of x. Equations like these are known as **differential equations**.

Example 11: The gradient of a tangent to the curve of $y = f(x)$ is $24x^2 + 10x$.
Express y in terms of x, given that the graph of $y = f(x)$ passes through the point (-1, -10).

Past Paper Example 1: Evaluate $\int_1^9 \frac{x+1}{\sqrt{x}}\,dx$

Past Paper Example 2: The parabola shown in the diagram has equation

$$y = 32 - 2x^2.$$

The shaded area lies between the lines $y = 14$ and $y = 24$.

Calculate the shaded area.