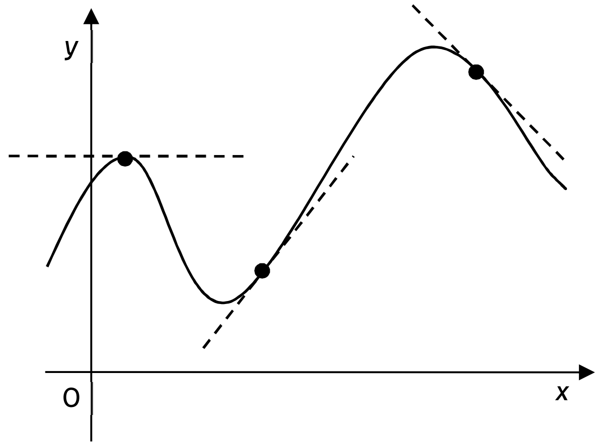
Differentiation

We know that the gradient of a straight line is a measure of how quickly it increases or decreases at a constant rate.

This is easy to see for linear functions, but what about quadratic, cubic and higher functions? As these functions produce curved graphs, they do **not** increase or decrease at a constant rate.

For a function f(x), the rate of change at any point on the function can be found by measuring the gradient of a **tangent** to the curve at that point.



The rate of change at any point of a function is called the **derived function** or the **derivative.**

Finding the rate of change of a function at a given point is part of a branch of maths known as **calculus.**

For function f(x) or the graph y = f(x), the derivative is written as:

f'(x) ("f dash x")

OR

$\dfrac{dy}{dx}$ **("dy by dx")**

Derivative = Rate of Change of the Function = Gradient of the Tangent to the Curve

The Derivative of $f(x) = ax^n$

Example 1: Find the derivative of $f(x) = x^2$

To find the derivative of a function:

1. **Make sure it's written in the form $y = ax^n$**
2. **Multiply by the power**
3. **Decrease the power by one**

Example 2: $f(x) = 2x^3$. Find $f'(x)$.

This means:

At $x = 1$, the gradient of the tangent to $2x^3$ =

At $x = -2$, the gradient of the tangent to $2x^3$ =

If $f(x) = ax^n$, then $f'(x) = nax^{n-1}$

The *DE* rivative *DE* creases the power!

To find the derivative of f(x):

- f(x) MUST be written in the form $f(x) = ax^n$
- Rewrite to eliminate fractions by using negative indices
- Rewrite to eliminate roots by using fractional indices

Revision from National 5

Example 3: Write with negative indices:

a) $\dfrac{2}{x^2}$ b) $\dfrac{1}{4x^5}$ c) $\dfrac{3}{5x}$

Example 4: Write in index form:

a) $\sqrt{x}$ b) $\sqrt[3]{x^2}$ c) $\dfrac{2}{3\sqrt{x^7}}$

Example 5: For each function, find the derivative.

a) $f(x) = x^{35}$

b) $g(x) = x^{-4}$ $(x \neq 0)$

c) $p(x) = -x^{-3}$ $(x \neq 0)$

d) $y = 12x^5 + 3x^2 - 2x + 9$

e) $y = \frac{1}{3\sqrt{x}}$ $(x > 0)$

f) $y = \left(\sqrt{x} - 2\right)^2$ $(x \geq 0)$

Example 6: Find the rate of change of each function:

a) $f(x) = \frac{x^5 - 3x}{2x^3}$ $(x \neq 0)$

b) $y = \frac{(x+3)^2}{x^{\frac{2}{3}}}$ $(x \neq 0)$

Points to note:	• **Number terms disappear (e.g. if f(x) = 5, f'(x) = 0)** • ***x* - terms leave their coefficient (e.g. if f(x) = 135x , f'(x) = 135)** • **Give your answer back in the same form as the question**

Equation of a Tangent to a Curve

Example 7: Find the equation of the tangent to the curve $y = x^2 - 2x - 15$ when $x = 4$.

To find the equation of a tangent to a curve:

- **Find the point of contact (sub the value of *x* into the equation to find *y*)**
- **Find $\frac{dy}{dx}$**
- **Find *m* by substituting *x* into $\frac{dy}{dx}$**
- **Use $y - b = m(x - a)$**

Example 8: f(x) is defined on the set of real numbers as $f(x) = x^3 - 2x^2$.

a) Find the gradient of the tangent to the curve $y = f(x)$ at the point where $x = -1$

b) Find the x-coordinate of the second point on the curve where the tangent has the same gradient.

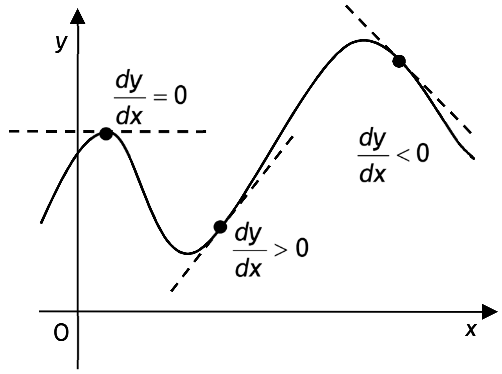
Increasing & Decreasing Functions

For any curve,

if $\frac{dy}{dx} > 0$, then y is **increasing**

if $\frac{dy}{dx} < 0$, then y is **decreasing**

if $\frac{dy}{dx} = 0$, then y is **stationary**



Example 9: State whether the function $f(x) = x^3 - x^2 - 5x + 2$ is **increasing**, **decreasing** or **stationary** when:

a) $x = 0$

b) $x = 1$

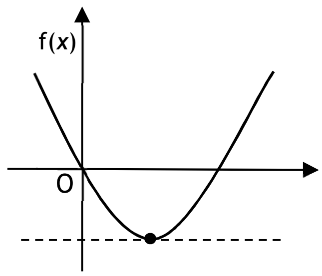
c) $x = 2$

Example 10: Show algebraically that the function $f(x) = x^3 - 6x^2 + 12x - 5$ is never decreasing.

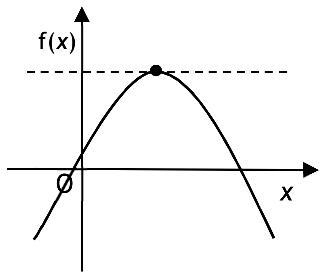
Example 11: Find the intervals in which the function $f(x) = 2x^3 - 6x^2 + 5$ is: (i) increasing
(ii) decreasing

Stationary Points and their Nature

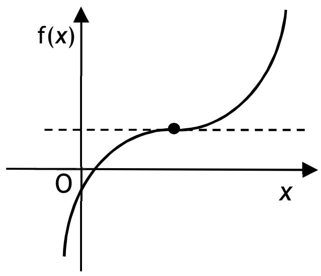
Any point on a curve where the tangent is horizontal (i.e. the gradient or $\frac{dy}{dx} = 0$) is commonly known as a **stationary point.** There are four types of stationary point:



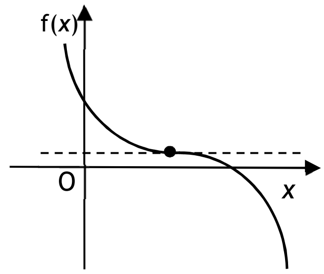
Minimum Turning Point



Maximum Turning Point



Rising Point of Inflection



Falling Point of Inflection

To locate the position of stationary points, we find the derivative, make it equal zero, and solve for x. To determine their type (or **nature**), we use a **nature table**.

Example 12: Find the stationary points of the curve $y = 2x^3 - 12x^2 + 18x$ and determine their nature.

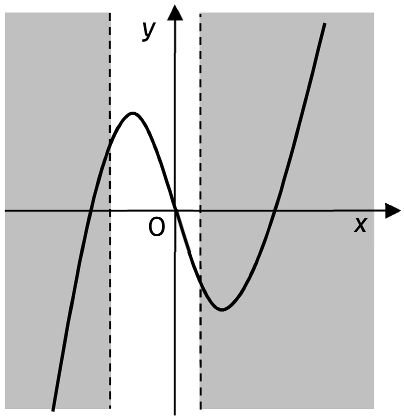
Watch out for:	**x as a factor**	e.g.	**$x(x - 3) = 0$**	leads to SPs at	**$x = 0$**	**&**	**$x = 3$**
	square roots	e.g.	**$x^2 = 4 = 0$**	leads to SPs at	**$x = 2$**	**&**	**$x = -2$**

Closed Intervals

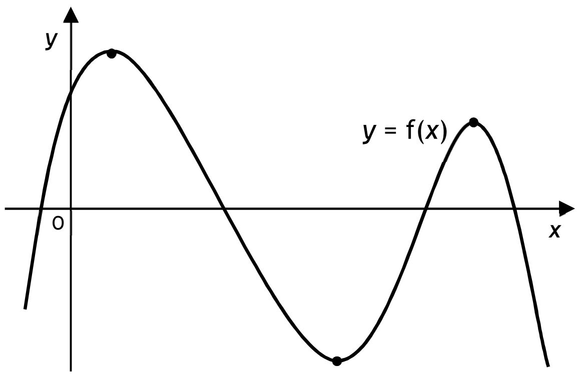
Sometimes, we may only be interested in a small section of the curve of a function. To find the maximum and minimum values of a function in a given interval, we find stationary points as normal, but we also need to consider the value of the function at the ends of the interval.

The nature of these points is not relevant, so a nature table is not required!

Example 13: Find the greatest and least values of y = $x^3 - 12x$ on the interval $-3 \le x \le 1$.



Graph of the Derived Function



From the graph of y = f(x), we can obtain the graph of y = f'(x) by considering its stationary points. On the graph of y = f'(x), the y-coordinate comes from the derivative of y = f(x).

Example 14: The graph of y = f(x) is shown. Sketch the graph of y = f'(x).

1. Draw a set of axes directly under a copy of y = f(x).

2. Locate the stationary points.

3. At SP's, f'(x) = 0, so the y coordinate of f'(x) = 0 on the new graph.

4. Where f(x) is increasing, f'(x) is **above** the x – axis.

5. Where f(x) is decreasing, f'(x) is **below** the x – axis.

6. Draw a smooth curve which fits this information.

Past Paper Example 1: A function f is defined on a suitable domain by $f(x) = \sqrt{x}\left(3x - \frac{3}{x\sqrt{x}}\right)$.

Find $f'(4)$.

Past Paper Example 2: A curve has equation $y = x^4 - 4x^3 + 3$. Find the position and nature of its stationary points.

Past Paper Example 3: Find the equations of the two tangents to the curve $y = 2x^3 - 3x^2 - 12x + 20$ which are parallel to the line $48x - 2y = 5$.