Sets and Functions

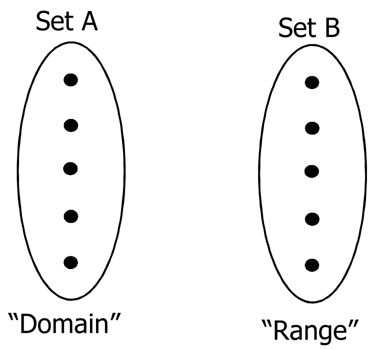
A **set** is a group of numbers which share common properties. Some common sets are:

Natural Numbers	N = {1, 2, 3, 4, 5,……}
Whole Numbers	W = {0, 1, 2, 3, 4, 5,…..}
Integers	Z = {…..,-3, -2, -1, 0, 1, 2, 3,…..}
Rational Numbers	Q = all integers **and** fractions of them (e.g. ¾, -⅝, etc)
Real Numbers	R = all rational **and** irrational numbers (e.g. $\sqrt{2}, \pi$, etc.)

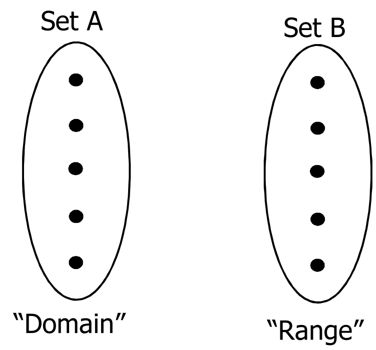
Sets are written inside curly brackets. The set with no members "{ }" is called the **empty set.**

$\in$ means "is a member of", e.g. $5 \in \{3, 4, 5, 6, 7\}$ $\notin$ means "is not a member of", e.g. $5 \notin \{6, 7, 8\}$

A **function** is a rule which links an element in Set A to **one and only one** element in Set B.



This shows a function



This does **not** show a function

The set that the function works on is called the **domain**; the values produced are called the **range.** For graphs of functions, we can think of the **domain** as the ***x* - values**, and the **range** as the ***y* - values**.

This means that any operation which produces **more than one answer** is **not** considered a function. For example, since √4 = 2 **and** -2, "f(x) = √x" is **not** considered a function.

Example 1: Each function below is defined on the set of real numbers. State the **range** of each.

a) $f(x) = \sin x°$

b) $g(x) = x^2$

c) $h(x) = 1 - x^2$

When choosing the domain, two cases MUST be avoided:	**a) Denominators can't be zero** **b) Can't find the square root of a negative value**

e.g. For $f(x) = \frac{1}{x+5}$, $x \neq -5$, i.e. $\{x \in R : x \neq -5\}$

e.g. For $g(x) = \sqrt{x-3}$, $x \geq 3$, i.e. $\{x \in R : x \geq 3\}$

Example 2: For each function, state a suitable domain.

a) $g(x) = \sqrt{3x-2}$

b) $p(\theta) = \frac{2}{5-\theta}$

c) $f(y) = \frac{y^2}{\sqrt{y-1}}$

Composite Functions

In the linear function $y = 3x - 5$, we get y by doing **two** acts: (i) multiply x by 3; (ii) then subtract 5. This is called a **composite function**, where we "do" a function to the range of another function.

e.g. If h(x) is the composite function obtained by performing f(x) on g(x), then we say

h(x) = f(g(x)) ("f of g of x")

Example 3: $f(x) = 5x + 1$ and $g(x) = 3x^2 + 2x$.

a) Find f(g(-1))

b) Find f(g(x))

c) Find f(f(x))

d) Find g(f(x))

NOTE: Usually, f(g(x)) and g(f(x)) are NOT the same!

Example 4: $f(x) = 2x + 1$, $g(x) = x^2 + 6$.

a) Find formulae for:

(i) f(g(x))

(ii) g(f(x))

b) Solve the equation f(g(x)) = g(f(x))

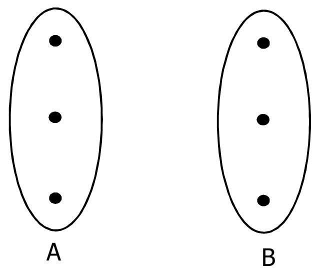
Example 5: $f(x) = \frac{3}{x+1}$, $x \neq -1$. Find an expression for f(f(x)) as a fraction in its simplest form.

Inverse Functions

If a function links every number in the domain to **only** one number in the range, the function is called a **one to one correspondence**.

When function f(x) is a one to one correspondence from A to B, the function which maps from B back to A is called the **inverse function**, written **$f^{-1}(x)$.**

For example, if f(x) = 2x, the inverse would be the function which "cancels out" multiplication by 2, i.e. $f^{-1}(x) = \frac{1}{2}x$



Finding the Formula of an Inverse Function

We can find the formula for the inverse of a function through a process very similar to changing the subject of a formula.

Example 6: For each function shown find a formula for $f^{-1}(x)$, the inverse function.

a) $f(x) = 2x + 5$

b) $f(x) = \frac{1}{2}(x - 9)$

c) $f(x) = 9 - \frac{x}{2}$

d) $f(x) = 3x^3 - 4$

If f(g(x)) = x , then f(x) and g(x) are inverse functions, so that

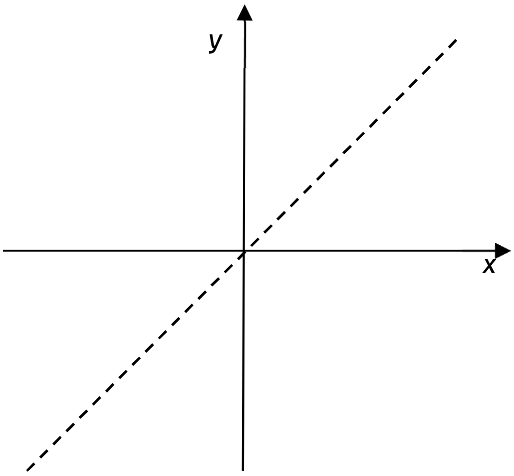
$f(x) = g^{-1}(x)$ AND $g(x) = f^{-1}(x)$

Example 7: $f(x) = 2x + 5$ and $g(x) = \frac{x-5}{2}$. Show that $g(x) = f^{-1}(x)$

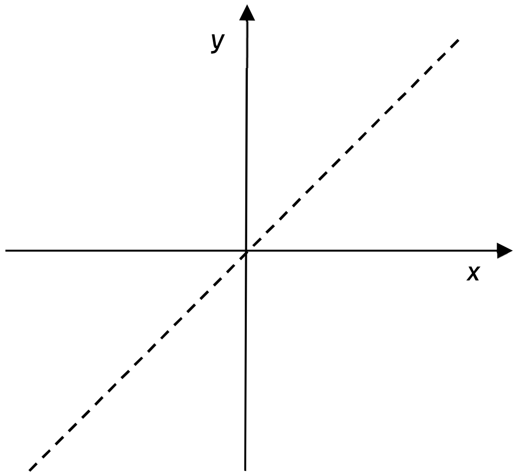
Graphs of Inverse Functions

Example 8: Using your answers to Example 5, sketch on the same graphs below:

a) $y = f(x)$ and $y = f^{-1}(x)$ where $f(x) = 2x + 5$



b) $y = f(x)$ and $y = f^{-1}(x)$ where $f(x) = \frac{x}{2} - 9$



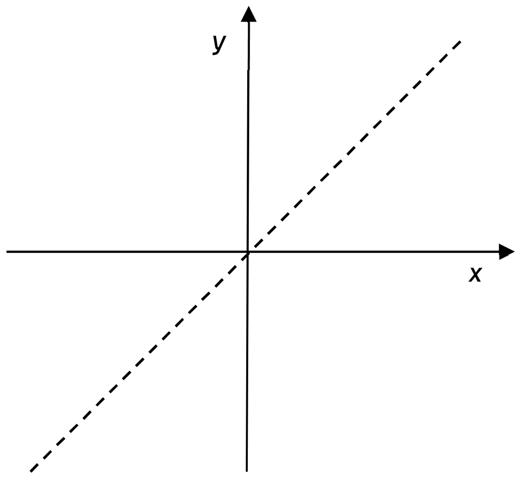
The dotted lines on each diagram are the line y = x. In each case, the graph of an inverse function can be obtained from the graph of the original function by **reflecting in the line y = x.**

Example 9: $g(x) = x^3 + 6$

a) Sketch the graph of $y = g(x)$.

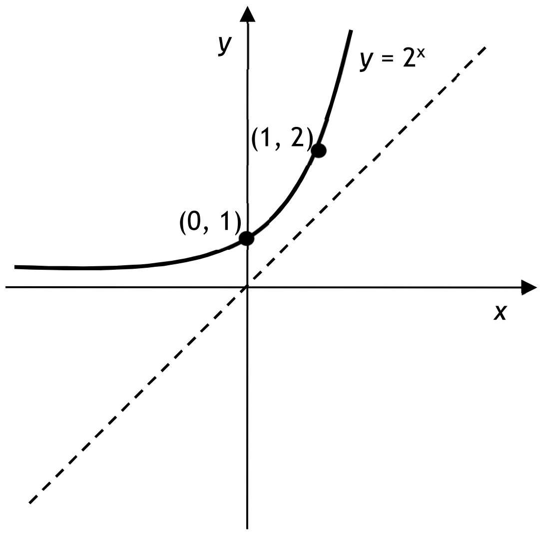
b) Show that $g^{-1}(x) = \sqrt[3]{x - 6}$

c) Hence sketch the graph of $y = \sqrt[3]{x - 6}$



Exponential and Logarithmic Functions

Exponential functions have the formula $f(x) = a^x$, $x \in \mathbf{R}$, where a is called the **base**. The graph of $y = 2^x$ is shown below.



The graph of $y = 2^x$ passes through the points (0, 1) and (1, 2). As reflection in the line $y = x$ will produce the inverse of $y = 2^x$, then the inverse of $f(x) = 2^x$ must pass through the points (1, 0) and (2, 1).

The inverse of an exponential function is known as a **logarithmic** function.

If $f(x) = a^x$, then $f^{-1}(x) = \log_a x$
("log to the base a of x")

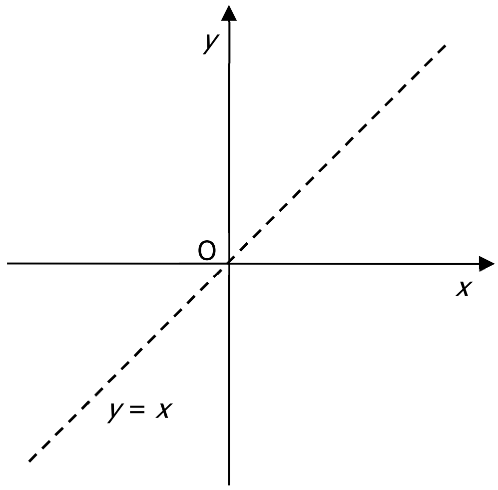
Example 10 Add the graph of $y = \log_2 x$ to the graph opposite.

Note that:

$y = a^x$ passes through (0, 1) and (1, a)
$y = \log_a x$ passes through (1, 0) and (a, 1)

For logarithms:

If $y = a^x$
then
$\log_a y = x$



Example 11 Sketch and annotate the graphs of:

a) $y = 5^x$

b) $y = \log_5 x$

Example 12 Write as logarithms:

a) $y = 3^x$

b) $q = 13^g$

$y = a^x$ means "a multiplied by itself x times gives y"

$\log_a x = y$ means "y is the number of times I multiply a by itself to get x"

Past Paper Example 1: Functions f and g are defined on the set of real numbers by

$f(x) = x^2 + 3$ $\qquad$ $g(x) = x + 4$

a) Find expressions for:

(i) $f(g(x))$

(ii) $g(f(x))$

b) Show that $f(g(x)) + g(f(x)) = 0$ has no real roots.

Past Paper Example 2: A function $g(x)$ is defined on the set of real numbers by $g(x) = 6 – 2x$.

a) Determine an expression for $g^{-1}(x)$.

b) Write down an expression for $g(g^{-1}(x))$.