Mini-Prelim Examination 2018 / 2019

MATHEMATICS

Advanced Higher Grade

Time allowed - 1 hour 20 minutes

Total marks – 44

Attempt ALL questions.

You may use a calculator.
Full credit will be given only to solutions which contain appropriate working.
State the units for your answer where appropriate.
Answers obtained by readings from scale drawings will not receive any credit.

Write your answers clearly in the answer booklet provided. In the answer booklet, you must clearly identify the question number you are attempting.
Use **blue** or **black** ink.
Before leaving the examination room you must give your answer booklet to the Invigilator; if you do not, you may lose all the marks for this paper.

FORMULAE LIST

Summations

(Arithmetic series) $S_n = \frac{1}{2}n[2a + (n-1)d]$

(Geometric series) $S_n = \frac{a(1-r^n)}{1-r}$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2},\ \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6},\ \sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

Binomial theorem

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r \quad \text{where } \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

Maclaurin expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \ldots$$

De Moivre's theorem

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Vector product

$$\mathbf{a}\times\mathbf{b} = |a||b|\sin\theta\hat{n} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

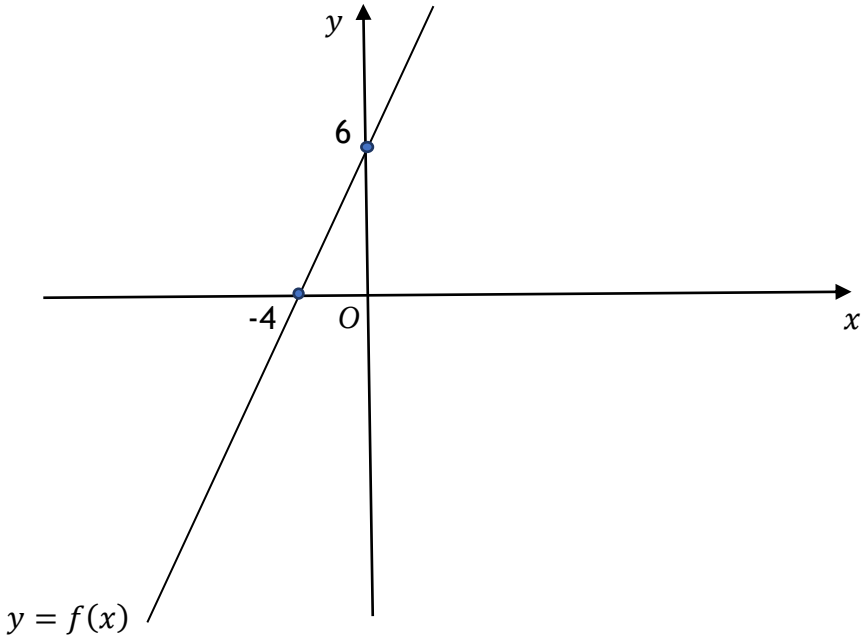
Matrix transformation

Anti-clockwise rotation through an angle, θ, about the origin, $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

[Turn over

Total marks – 44
Attempt ALL questions.

1. Shown in the diagram below is part of the graph with equation $y = f(x)$.



Sketch the graph of the function with equation $y = |6 - f(x)|$. **2**

2. (a) Given that $z = 1 + 3i$ is a root of the equation $z^3 - 4z^2 + \alpha z - 20 = 0$ $(\alpha \in \mathbb{R})$, find the value of α. **3**

(b) State a second root of the equation $z^3 - 4z^2 + \alpha z - 20 = 0$. **1**

(c) Hence find all of the roots of the equation $z^3 - 4z^2 + \alpha z - 20 = 0$. **4**

3. (a) Find the first four terms in the Maclaurin expansion of

$$f(x) = \frac{1}{\sqrt{4-x}}\,.$$ **4**

(b) **Hence** the first four terms in the Maclaurin expansion of

$$g(x) = \frac{1}{\sqrt{4-x}} + \frac{1}{\sqrt{4-2x}}.$$ **2**

[Turn over for Questions 4, 5, 6, 7 & 8

4. Find the coefficient of x^5 in the expansion of $\left(x^3+\frac{2}{x}\right)^7$. **3**

5. A geometric series has first term $\frac{1}{k}$ and fourth term $\sqrt{k}$ where k is a natural number.

(a) Find, in terms of k, an expression for the common ratio. **2**

(b) Given that the seventh term of the series is 4, find the value of k. **2**

(c) S_n denotes the sum to n terms of the series.

Given that $S_n - S_{n-1} = 2$, find the value of n. **3**

6. The area of a circle is increasing at a steady rate of $1.6\text{cm}^2\text{s}^{-1}$.

Find the rate at which the radius of the circle is increasing when the area of the circle is 3cm^2. **4**

7. Prove by induction that

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$$

for all natural numbers n. **5**

8. The lines L_1 and L_2 have equations

$$L_1: \frac{x-9}{1} = \frac{y-13}{4} = \frac{z+3}{-2} \ (= t_1)$$

$$L_2: \frac{x-2}{2} = \frac{y+1}{1} = \frac{z-1}{1} \ (= t_2).$$

(a) The lines L_1 and L_2 intersect at the point Q.

Find the coordinates of the point Q. **3**

(b) Find the size of the acute angle between the lines L_1 and L_2. **2**

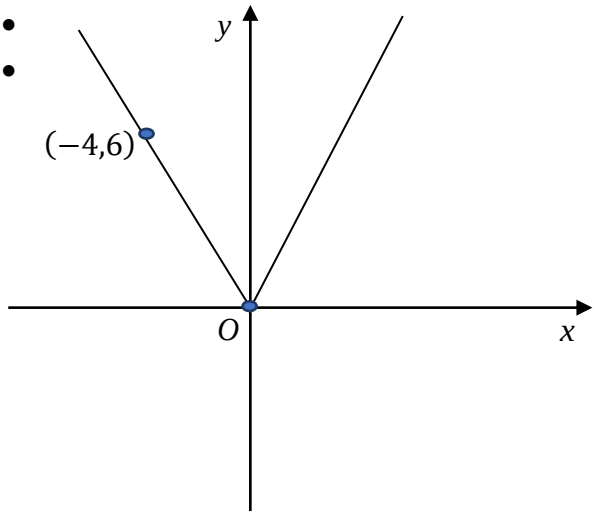
(c) The point R has coordinates $(4, 16, -3)$.

The point P lies on the line L_1 such that PR is perpendicular to L_1.

Find the coordinates of the point P. **4**

[*END OF QUESTION PAPER*]

Marking Scheme – Advanced Higher Grade 2018 / 2019
Mini-Prelim

	Give one mark for each •	Illustrations for awarding each mark
1	**ans: *Check graph*** **2 marks** • correct shape of graph • correct points annotated	•  •
2(a)	**ans: $\alpha = 14$** **3 marks** • substitutes correctly • simplifies correctly • correct answer	• $(1+3i)^3 - 4(1+3i)^2 + \alpha(1+3i) - 20 = 0$ • $1 + 3(3i) + 3(3i)^2 + (3i)^3 - 4(1+6i-9) + \alpha + 3\alpha i - 20 = 0$ $\Rightarrow 1 + 9i - 27 - 27i - 4 - 24i + 36 + \alpha + 3\alpha i - 20 = 0$ $\Rightarrow (\alpha - 14) + (3\alpha - 42)i = 0$ • $\Rightarrow \alpha - 14 = 0$ $\Rightarrow \alpha = 14$
2(b)	**ans: $(z =)\ 1 - 3i$** **1 mark** • correct root stated	• $1 - 3i$
2(c)	**ans: $(z =)\ 2, (1 + 3i\ \&1 - 3i)$** **4 marks** • correct quadratic factor • correct strategy; i.e. algebraic long division (or equivalent) • correct quotient • correct root	• $\big(z - (1+3i)\big)\big(z - (1-3i)\big)$ $= z^2 - (1-3i)z - (1+3i)z + (1+3i)(1-3i)$ $= z^2 - 2z + 10$ • $z^2 - 2z + 10 / z^3 - 4z^2 + 14z - 20$ • $z - 2$ • 2

	Give one mark for each •	Illustrations for awarding each mark
3(a)	**ans:** $\frac{1}{2}+\frac{1}{16}x+\frac{3}{256}x^2+\frac{5}{2048}x^3$ **4 marks** • starts correctly • continues correctly • substitutes correctly • correct simplified answer	• $f(x)=\frac{1}{\sqrt{4-x}} \Rightarrow f(0)=\frac{1}{2}$ & $f'(x)=\frac{1}{2\sqrt{4-x}^3} \Rightarrow f'(0)=\frac{1}{16}$ • $f''(x)=\frac{3}{4\sqrt{4-x}^5} \Rightarrow f''(0)=\frac{3}{128}$ & $f'''(x)=\frac{15}{8\sqrt{4-x}^7} \Rightarrow f'''(0)=\frac{15}{1024}$ • $\frac{1}{2}+\frac{\frac{1}{16}}{1!}x+\frac{\frac{3}{128}}{2!}x^2+\frac{\frac{5}{2048}}{3!}x^3$ • $\frac{1}{2}+\frac{1}{16}x+\frac{3}{256}x^2+\frac{5}{2048}x^3$
3(b)	**ans:** $1+\frac{3}{16}x+\frac{15}{256}x^2+\frac{45}{2048}x^3$ **2 marks** • substitutes correctly • simplifies correctly	• $\ldots+\frac{1}{2}+\frac{1}{16}(2x)+\frac{3}{256}(2x)^2+\frac{5}{2048}(2x)^3$ • $\frac{1}{2}+\frac{1}{16}x+\frac{3}{256}x^2+\frac{5}{2048}x^3+$ $\frac{1}{2}+\frac{1}{16}(2x)+\frac{3}{256}(2x)^2+\frac{5}{2048}(2x)^3$ $=1+\frac{3}{16}x+\frac{15}{256}x^2+\frac{45}{2048}x^3$
4	**ans:** ***560*** **3 marks** • correct general term • put power of x equal 5 and solve for r • calculate coefficient	• $\binom{7}{r}\left(x^3\right)^{7-r}\left(\frac{2}{x}\right)^r=\binom{7}{r}2^r x^{21-4r}$ • $21-4r=5;\ r=4$ • $\binom{7}{4}2^4=35\times16=560$
5(a)	**ans:** $r=\sqrt{k}$ **2 marks** • substitutes & starts to solve correctly • solves correctly	• $a=\frac{1}{k}$ & $ar^3=\sqrt{k} \Rightarrow \frac{1}{k}r^3=\sqrt{k} \Rightarrow r^3=k^{\frac{3}{2}}$ • $r=k^{\frac{1}{2}}$
5(b)	**ans:** $k=2$ **2 marks** • substitutes correctly • solves correctly	• $ar^6=4 \Rightarrow \frac{1}{k}\left(\sqrt{k}\right)^6=4$ • $\Rightarrow \frac{1}{k}.k^3=4 \Rightarrow k^2=4 \Rightarrow k=\pm2 \Rightarrow k=2$
5(c)	**ans:** $n=5$ **3 marks** • correct expression for sum to n terms or first three sums correct • correct strategy: starts to find differences of consecutive sums • correct answer (with justification)	• $S_n=\frac{\frac{1}{2}\left(1-\sqrt{2}^n\right)}{1-\sqrt{2}}$ ***or*** $S_1=\frac{1}{2}, S_2=\frac{1}{2}+\frac{1}{\sqrt{2}}, S_3=\frac{3}{2}+\frac{1}{\sqrt{2}}$ • $S_2-S_1=\frac{1}{2}+\frac{1}{\sqrt{2}}-\frac{1}{2}=\frac{1}{\sqrt{2}}$, $S_3-S_2=\frac{3}{2}+\frac{1}{\sqrt{2}}-\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right)=1$ • $S_4-S_3=\frac{3}{2}+\frac{3}{\sqrt{2}}-\left(\frac{3}{2}+\frac{1}{\sqrt{2}}\right)=\frac{2}{\sqrt{2}}=\sqrt{2}$,

		$S_5 - S_4 = \frac{7}{2} + \frac{3}{\sqrt{2}} - \left(\frac{3}{2} + \frac{3}{\sqrt{2}}\right) = 2 \Rightarrow n = 5$
	Give one mark for each •	**Illustrations for awarding each mark**
6	**ans:** $\frac{dr}{dt} = 0.26\ \mathbf{cms^{-1}}$ **4 marks** • states and differentiates correct relationship • correct value for r • starts to substitute into correct relationship • completes substitution and evaluates correctly	• $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$ • $A = 3 \Rightarrow \pi r^2 = 3 \Rightarrow r^2 = \frac{3}{\pi} \Rightarrow r = \pm\sqrt{\frac{3}{\pi}}$ $\Rightarrow r = \sqrt{\frac{3}{\pi}}$ • $\frac{dA}{dt} = \frac{dA}{dr}.\frac{dr}{dt} \Rightarrow 1.6 = 2\pi r.\frac{dr}{dt}$ • $\Rightarrow \frac{dr}{dt} = \frac{1.6}{2\pi r} \Rightarrow \frac{dr}{dt} = \frac{1.6}{2\pi\sqrt{\frac{3}{\pi}}} \Rightarrow \frac{dr}{dt} = 0.26\text{cms}^{-1}$
7	**ans:** ***Proof*** **5 marks** • knows how to start proof: proves result is true for $n = 1$ • states correct assumption for $n = k$ • states correct requirement for $n = k + 1$ • continues proof correctly • completes proof correctly	• Let $n = 1$: LHS $= \frac{2(1)+1}{1^2(1+1)^2} = \frac{3}{4}$ & RHS $= 1 - \frac{1}{(1+1)^2} = \frac{3}{4} \Rightarrow$ result is true for $n = 1$ • Assume result is true for $n = k$: $\sum_{r=1}^{k} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(k+1)^2}$ • Required to prove result is true for $n = k + 1$: $\sum_{r=1}^{k+1} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(k+2)^2}$ • $\sum_{r=1}^{k+1} \frac{2r+1}{r^2(r+1)^2} = \sum_{r=1}^{k} \frac{2r+1}{r^2(r+1)^2} + \frac{2(k+1)+1}{(k+1)^2(k+2)^2}$ $= 1 - \frac{1}{(k+1)^2} + \frac{2k+3}{(k+1)^2(k+2)^2}$ $= 1 - \left\{\frac{1}{(k+1)^2} - \frac{2k+3}{(k+1)^2(k+2)^2}\right\}$ $= 1 - \frac{(k+2)^2 - (2k+3)}{(k+1)^2(k+2)^2}$ • $= 1 - \frac{k^2+2k+1}{(k+1)^2(k+2)^2}$ $= 1 - \frac{(k+1)^2}{(k+1)^2(k+2)^2}$ $= 1 - \frac{1}{(k+2)^2}$ ∴ Since true for

		$n = 1$ and $\begin{pmatrix}\text{true for } n = k \Rightarrow \\ \text{true for } n = k+1\end{pmatrix}$, the result is true for all natural numbers
	Give one mark for each •	**Illustrations for awarding each mark**
8(a)	**ans: $\boldsymbol{Q(6,1,3)}$ 3 marks** • correct strategy: starts to solve set of three equations in two variables • solves correctly • correct coordinates	• $t_1 - 2t_2 = -7$ $4t_1 - t_2 = -14$ $2t_1 + t_2 = -4$ • $t_1 = -3, t_2 = 2$ • $x = 6, y = 1, z = 3$
8(b)	**ans: $\boldsymbol{69.1^\circ}$ 2 marks** • knows how to find angle between planes • calculates angle between planes correctly	• $\cos\theta = \frac{\begin{pmatrix}1\\4\\-2\end{pmatrix}.\begin{pmatrix}2\\1\\1\end{pmatrix}}{\sqrt{21}\sqrt{6}}$ • 69.1° (or 1.206 radians)
8(c)	**ans: $\boldsymbol{P\left(\frac{28}{3}, \frac{43}{3}, -\frac{11}{3}\right)}$ 4 marks** • starts a correct strategy; e.g. finds $\overrightarrow{PR}$ • continues a correct strategy; e.g. sets $\overrightarrow{PR}.\boldsymbol{a} = 0$ and starts to simplify • calculates value of t_1 correctly • correct coordinates	• $\overrightarrow{PR} = \begin{pmatrix}-5 - t_1\\3 - 4t_1\\2t_1\end{pmatrix}$ • $\overrightarrow{PR}.\boldsymbol{a} = 0 \Rightarrow \begin{pmatrix}-5 - t_1\\3 - 4t_1\\2t_1\end{pmatrix}.\begin{pmatrix}1\\4\\-2\end{pmatrix} = 0$ $\Rightarrow 1(-5 - t_1) + 4(3 - 4t_1) - 2(2t_1) = 0$ • $\Rightarrow t_1 = \frac{1}{3}$ • $x = \frac{28}{3}, y = \frac{43}{3}, z = -\frac{11}{3}$