1st Prelim Examination 2018 / 2019

MATHEMATICS

Advanced Higher Grade

Time allowed - 2 hours

Total marks - 65

Attempt ALL questions.

You may use a calculator.
Full credit will be given only to solutions which contain appropriate working.
State the units for your answer where appropriate.
Answers obtained by readings from scale drawings will not receive any credit.

Write your answers clearly in the answer booklet provided. In the answer booklet, you must clearly identify the question number you are attempting.
Use **blue** or **black** ink.
Before leaving the examination room you must give your answer booklet to the Invigilator; if you do not, you may lose all the marks for this paper.

FORMULAE LIST

Standard derivatives	
$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

Standard integrals	
$f(x)$	$\int f(x)dx$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax)+c$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)+c$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$
$\frac{1}{x}$	$\ln\lvert x\rvert+c$
e^{ax}	$\frac{1}{a}e^{ax}+c$

Vector product

$$\mathbf{a}\times\mathbf{b} = |a||b|\sin\theta\hat{n} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Matrix transformation

Anti-clockwise rotation through an angle, θ, about the origin, $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

[Turn over

Total marks – 65
Attempt ALL questions.

1. Find and simplify $f'(x)$ where:

(a) $f(x) = \sqrt{x}e^{-x}$ **4**

(b) $f(x) = \frac{10-4x}{1+e^{2x}}$ **3**

(c) Given $y = 3^x$, use logarithmic differentiation to obtain $\frac{dy}{dx}$ in terms of x. **3**

2. Use Gaussian elimination to solve the system of equations:

$$\begin{aligned} x + y + 3z &= 2 \\ 2x + y + z &= 2 \\ 3x + 2y + 5z &= 5 \end{aligned}$$ **5**

3. Use the division algorithm to express 1234_5 in base 7. **4**

4. Use the Euclidean algorithm to find d, the greatest common divisor of 714 and 582.

Hence find integers u and v such that $d = 714u + 582v$. **4**

5. A curve C has equation:

$$x^3 + 4xy - 2x - 2y^3 + 3 = 0.$$

(a) Use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y. **3**

(b) Find the equation of the tangent to the curve C at the point $(1, -1)$. **2**

6. Integrate $x^2 \ln x$ with respect to x. **4**

[Turn over for Questions 7, 8, 9, & 10

7. The matrix M is given by $M = \begin{pmatrix} x & 2 \\ 4 & -1 \end{pmatrix}$ where x is an integer.

(a) Given that $M^2 = 2I - 7M$, where I is the 2×2 identity matrix, find the value of x. **3**

(b) **Hence** find rational numbers a and b such that $M^{-1} = aI + bM$. **2**

(c) The transformation represented by the matrix M maps the point $P(x, y)$ to the point $Q(2 - 2t, t + 1)$ where t is an integer.

Find, in terms of t, the coordinates of the point P. **3**

8. Use the substitution $u = 1 + 3\tan x$ to evaluate

$$\int_0^{\frac{\pi}{4}} \frac{\sqrt{1+3\tan x}}{\cos^2 x}\, dx\,.$$ **5**

9. (a) Express $\frac{x+2}{(1-2x)(1+x^2)}$ in the form $\frac{a}{1-2x} + \frac{bx+c}{1+x^2}$ where a, b and c are constants. **3**

(b) Hence find the particular solution of the differential equation

$$\frac{dy}{dx} = \frac{x+2}{e^{2y}(1-2x)(1+x^2)}$$

given that $y = 0$ when $x = 0$.

Express your answer for y in terms of x. **7**

10. Find the particular solution of the equation $\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 55\sin x - 10\cos x$ given that $y = 4$ and $\frac{dy}{dx} = -29$ when $x = 0$. **10**

[*END OF QUESTION PAPER*]

Marking Scheme – Advanced Higher Grade 2018 / 2019 1st Prelim

	Give one mark for each •	**Illustrations for awarding each mark**
1(a)	**ans:** $\frac{1}{2\sqrt{x}}e^{-x}(1-2x)$ **4 marks** • knows, and starts, to use product rule for differentiation correctly • one term differentiated correctly • completes differentiation correctly • factorised form	• $f'(x) = u'v + uv'$ • $\frac{1}{2}x^{-\frac{1}{2}}e^{-x}$ ***or*** $x^{\frac{1}{2}}(-1)e^{-x}$ • $\frac{1}{2}x^{-\frac{1}{2}}e^{-x} + x^{\frac{1}{2}}(-1)e^{-x}$ • $\frac{1}{2\sqrt{x}}e^{-x}(1-2x)$
1(b)	**ans:** $\frac{4(2xe^{2x}-6e^{2x}-1)}{(1+e^{2x})^2}$ (or equivalent) **3 marks** • knows, and starts, to use quotient rule for differentiation correctly • completes differentiation correctly • simplifies correctly	• $\frac{-4(1+e^{2x})-\cdots}{(1+e^{2x})^2}$ or $\frac{\ldots-(10-4x).2e^{2x}}{(1+e^{2x})^2}$ • $\frac{-4(1+e^{2x})-(10-4x).2e^{2x}}{(1+e^{2x})^2}$ • $\frac{-4-24e^{2x}+8xe^{2x}}{(1+e^{2x})^2}$ (or a **simplified** equivalent)
1(c)	**ans:** $ln3.3^x$ (or equivalent) **3 marks** • knows, and starts, to use logarithmic differentiation correctly • completes differentiation correctly • simplifies correctly	• $lny = xln3$ • $\frac{1}{y}\frac{dy}{dx} = ln3$ • $\frac{dy}{dx} = ln3.y = ln3.3^x$ (or a **simplified** equivalent)
2	**ans:** $(2,-3,1)$ (or equivalent) **5 marks** • correct augmented matrix and first modified system correct • second modified system correct • third modified system correct •• correct solution	•• $\begin{pmatrix}1&1&3&2\\2&1&1&2\\3&2&5&5\end{pmatrix} \sim \begin{pmatrix}1&1&3&2\\0&-1&-5&-2\\0&-1&-4&-1\end{pmatrix}$ (or equivalent) • $\sim \begin{pmatrix}1&1&3&2\\0&-1&-5&-2\\0&0&-1&-1\end{pmatrix}$ •• $\Rightarrow z = 1, y = -3, x = 2$

	Give one mark for each •	Illustrations for awarding each mark
3	**ans: 365_7** **4 marks** • Correct method to convert to base 10 • Converts to base 10 • Correct method to convert to base 7 • Correct answer	 • $1\times5^3+2\times5^2+3\times5^1+4\times5^0$ • 194 • $194\div7=27r5$ $27\div7=3r6$ $3\div7=0r3$ • 365_7
4	**ans: $d=6; u=-22$ & $v=27$** **4 marks** • starts correctly • continues to find d correctly • continues correctly • correct answer	 • $714=1\times582+132$ $582=4\times132+54$ • $132=2\times54+24$ $54=2\times24+6$ $24=4\times6+0$ $\Rightarrow d=6$ • $6=54-2\times24$ $=5\times54-2\times132$ $=5\times582-22\times132$ • $=27\times582-22\times714$ $\Rightarrow u=-22$ & $v=27$ (accept $(6=)\ 27\times582-22\times714$)
5(a)	**ans: $\frac{dy}{dx}=\frac{2-3x^2-4y}{4x-6y^2}$** **3 marks** • starts differentiation correctly • completes differentiation correctly • rearranges correctly	 • $3x^2+4y+4x\frac{dy}{dx}\ldots$ • $\ldots-2-6y^2\frac{dy}{dx}=0$ • $\Rightarrow(4x-6y^2)\frac{dy}{dx}=2-3x^2-4y$ $\Rightarrow\frac{dy}{dx}=\frac{2-3x^2-4y}{4x-6y^2}$
5(b)	**ans: $3x+2y-1=0$** (or equivalent) **2 marks** • correct gradient • correct equation	 • $\frac{dy}{dx}=\frac{2-3+4}{4-6}\Rightarrow m=-\frac{3}{2}$ • $y+1=-\frac{3}{2}(x-1)\Rightarrow y=-\frac{3}{2}x+\frac{1}{2}$
6	**ans: $\frac{1}{3}x^3\ln x-\frac{1}{9}x^3+C$** **4 marks** • Knows to use integration by parts • Applies integration by parts correctly • integrates correctly • Correct answer	 • $u=\ln x\Rightarrow u'=\frac{1}{x}\quad v'=x^2\Rightarrow v=\frac{1}{3}x^3$ • $\frac{1}{3}x^3\ln x-\int\frac{1}{x}\frac{1}{3}x^3dx=\frac{1}{3}x^3\ln x-\int\frac{1}{3}x^2dx$ • $\frac{1}{3}x^3\ln x-\frac{1}{9}x^3$ • $\frac{1}{3}x^3\ln x-\frac{1}{9}x^3+C$

	Give one mark for each •	Illustrations for awarding each mark
7(a)	**ans: $x = -6$** **3 marks** • squares matrix M correctly • finds matrix $2I - 7M$ correctly • equates and solves correctly	• $M^2 = \begin{pmatrix} x & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x & 2 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} x^2+8 & 2x-2 \\ 4x-4 & 9 \end{pmatrix}$ • $2I - 7M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 7x & 14 \\ 28 & -7 \end{pmatrix} = \begin{pmatrix} 2-7x & -14 \\ -28 & 9 \end{pmatrix}$ • $2x - 2 = -14 \Rightarrow x = -6$, $4x - 4 = -28 \Rightarrow x = -6$ ***or*** $x^2 + 8 = 2 - 7x \Rightarrow x = -1, -6 \Rightarrow x = -6$
7(b)	**ans: $a = \frac{7}{2}, b = \frac{1}{2}$** **2 marks** • pre-multiplies by M^{-1} and starts to simplify correctly • correct values for a and b	• $M^2 = 2I - 7M \Rightarrow M^{-1}M^2 = M^{-1}(2I - 7M) \Rightarrow M = 2M^{-1} - 7I$ • $a = \frac{7}{2}, b = \frac{1}{2}$ (accept $M^{-1} = \frac{7}{2}I + \frac{1}{2}M$)
7(c)	**ans: $P(2, 7 - t)$** **3 marks** • correct matrix equation • multiplies matrices and starts to solve correctly • solves equations to find correct coordinates	• $MP = Q \Rightarrow \begin{pmatrix} -6 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2-2t \\ t+1 \end{pmatrix}$ • $\Rightarrow \begin{pmatrix} -6x+2y \\ 4x-y \end{pmatrix} = \begin{pmatrix} 2-2t \\ t+1 \end{pmatrix}$ $\Rightarrow -6x + 2y = 2 - 2t, 4x - y = t + 1$ • $x = 2, y = 7 - t$
8	**ans: $\frac{14}{9}$** **5 marks** • differentiates and rearranges correctly • substitutes and integrates correctly • changes limits correctly • substitutes correctly • correct answer	• $u = 1 + 3\tan x \Rightarrow du = 3\sec^2 x\, dx$ $\Rightarrow du = \frac{3}{\cos^2 x}dx \Rightarrow \frac{1}{3}du = \frac{1}{\cos^2 x}dx$ (stated or implied) • $\frac{1}{3}\int \sqrt{u}\, du = \frac{1}{3}\left[\frac{2}{3}u^{\frac{3}{2}}\right]$ • $x = 0 \Rightarrow u = 1$ & $x = \frac{\pi}{4} \Rightarrow u = 4$ • $\Rightarrow \frac{1}{3}\left[\left(\frac{2}{3}(4)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(1)^{\frac{3}{2}}\right)\right]$ • $\frac{14}{9}$

	Give one mark for each •	Illustrations for awarding each mark
9(a)	**ans:** $\frac{2}{1-2x}+\frac{x}{1+x^2}$ **3 marks** • starts correctly • a correct constant • all constants correct	 • $x+2\equiv a(1+x^2)+(bx+c)(1-2x)$ • $a=2, b=1$ ***or*** $c=0$ • $a=2, b=1$ ***and*** $c=0$
9(b)	**ans:** $\boldsymbol{y=\frac{\ln\left(\ln\left\lvert\frac{1+x^2}{(1-2x)^2}\right\rvert+1\right)}{2}}$ (or equivalent) • starts correctly – rearranges and knows to integrate • substitutes correctly • integrates correctly • substitutes correctly • evaluates the constant of integration correctly • substitutes and prepares to express y as a function of x • expresses y as a function of x correctly (ignore the omission of modulus signs)	**7 marks** • $\int e^{2y}dy=\int\frac{x+2}{(1-2x)(1+x^2)}dx$ • $\int e^{2y}dy=\int\left(\frac{2}{1-2x}+\frac{x}{1+x^2}\right)dx$ • $\frac{e^{2y}}{2}=-\ln\lvert1-2x\rvert+\frac{1}{2}\ln\lvert1+x^2\rvert+c$ • $\frac{e^0}{2}=-\ln\lvert1\rvert+\frac{1}{2}\ln\lvert1\rvert+c$ • $c=\frac{1}{2}$ • $e^{2y}=2\left(\ln\left\lvert(1+x^2)^{\frac{1}{2}}\right\rvert-\ln\lvert1-2x\rvert+\frac{1}{2}\right)$ $\Rightarrow e^{2y}=2\ln\left\lvert\frac{(1+x^2)^{\frac{1}{2}}}{1-2x}\right\rvert+1$ • $y=\frac{\ln\left(2\ln\left\lvert\frac{\sqrt{1+x^2}}{1-2x}\right\rvert+1\right)}{2}$ (or equivalent)
10	**ans:** $y=\frac{9}{2}e^{-7x}-\frac{11}{2}xe^{-7x}+\sin x-\frac{1}{2}\cos x$ • solves auxiliary equation correctly • correct complementary function • correct form for particular integral • differentiates correctly • substitutes correctly • solves correctly • states correct general solution • substitutes and solves correctly • differentiates correctly • substitutes and solves to find correct particular solution	**10 marks** • $m^2+14m+49=0$ $\Rightarrow(m+7)(m+7)=0$ $\Rightarrow m=-7$(twice) • $y=Ae^{-7x}+Bxe^{-7x}$ • $y=C\sin x+\mathrm{D}\cos x$ • $\Rightarrow\frac{dy}{dx}=C\cos x-D\sin x$ $\Rightarrow\frac{d^2y}{dx^2}=-C\sin x-\mathrm{D}\cos x$ • $(48C-14D)\sin x+(14C+48D)\cos x=55\sin x-10\cos x$ • $\Rightarrow 48C-14D=55\Rightarrow 14C+48D=-10$ $\Rightarrow D=-\frac{1}{2}\Rightarrow C=1$ • $\therefore y=Ae^{-7x}+Bxe^{-7x}+\sin x-\frac{1}{2}\cos x$ • $x=0, y=4\Rightarrow A=\frac{9}{2}$ • $\frac{dy}{dx}=-7Ae^{-7x}+Be^{-7x}-7Bxe^{-7x}$ $+\cos x+\frac{1}{2}\sin x$ • $x=0, \frac{dy}{dx}=-29\Rightarrow B=-\frac{11}{2}$

TOTAL MARKS = 65