Unit 2: Applications in Algebra and Calculus (H7X1 77)

Applying Algebraic Skills to Summation and Mathematical Proof

Proof by Induction: Statement is made about . . .

- Shown that it works for an initial value $n = a$ (usually $n = 1$)
- Next assume statement is true for $n = k$ and get an expression
- Prove that it is true for $n = k + 1$ using algebraic manipulation.
- Conclude that if true for $n = k$ then true for $n = k + 1$ and since true for $n = a$, then true for all n.

Summation Formulae (Series):

❶ Prove by induction that $1 + 2 + 3 + \cdots = \frac{n}{2}(n+1)$ i.e.

$$\sum_{r=1}^{n} r = \frac{n}{2}(n+1), \quad n \geq 1, n \in \mathbb{N}$$

Step 1: Let $n = 1$: $\sum_{r=1}^{1} r = \frac{1}{2}(1+1) = 1$ ✓ *True*

Step 2: Assume true for $n = k$, expression is: $\sum_{r=1}^{k} r = \frac{k}{2}(k+1)$

Step 3: Get the similar expression for $k + 1$:

$$\sum_{r=1}^{k+1} r = \frac{k+1}{2}(k+1+1) = \frac{1}{2}(k+1)(k+2)$$ *This answer is our target!*

Step 4: Get alternative expression for $k + 1$ using answer to Step 2 plus $(k+1)^{st}$ term and show they are the same:

$$\sum_{r=1}^{k} r + (k+1) = \frac{k}{2}(k+1) + (k+1) = [k+1]\left(\frac{k}{2} + 1\right) = \frac{1}{2}(k+1)(k+2)$$ ✓

Step 5: Write a conclusion:

If true for $n = k$ then true for $n = k + 1$ and since true for $n = 1$,

then by induction, it is true $\forall n \geq 1, n \in \mathbb{N}$.

❷ Prove by induction that $1 + 4 + 9 + 16 + \cdots = \frac{n(n+1)(2n+1)}{6}$ i.e.

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}, \quad n \geq 1, n \in \mathbb{N}$$

Step 1: Let $n = 1$: $\sum_{r=1}^{1} r^2 = \frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1$ ✓ *True*

Step 2: Assume true for $n = k$: $\sum_{r=1}^{k} r^2 = \frac{k(k+1)(2k+1)}{6}$

Step 3: Get similar expression for $k + 1$:

$$\sum_{r=1}^{k+1} r = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} = \frac{1}{6}(k+1)(k+2)(2k+3)$$

Step 4: Get alternative expression for $k + 1$ using answer to 2 plus $(k+1)^{st}$ term and show that they the are the same:

$$\sum_{r=1}^{k} r^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \left[\frac{k+1}{6}\right][k(2k+1) + 6(k+1)]$$

$$= \left[\frac{k+1}{6}\right]\left[2k^2 + k + 6k + 6\right] = \frac{1}{6}(k+1)\left(2k^2 + 7k + 6\right)$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3) \quad ✓$$

NB – Algebra usually only worth 1 mark so if it's not working out then have a Eureka moment as the last mark is normally the conclusion!!

Step 5: Write conclusion:

If true for $n = k$ then true for $n = k + 1$ and since true for $n = 1$,

then by induction, it is true $\forall n \geq 1, n \in \mathbb{N}$.

Use proof by induction to show that $\forall n \geq 1, n \in \mathbb{N}$:

1. $\sum_{r=1}^{n}(3r-1) = \frac{n}{2}(3n+1)$

2. $\sum_{r=1}^{n}(2r+1) = n(n+2)$

3. $\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$

4. $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$

Multiple of/Divisible by:

❸ Prove by induction that $n^3 + 2n$ is divisible by 3, $\forall n \in \mathbb{N}$.

Step 1: Let $n = 1$: $n^3 + 2n = 1^3 + 2 \times 1 = 3, \quad 3|3$ ✓ *True*

Step 2: Assume true for $n = k$ *and make expression a multiple of 3:*

$$3|k^3 + 2k \Rightarrow k^3 + 2k = 3m, m \in \mathbb{N}$$

Step 3: Get expression for $k+1$:

$$n^3 + 2n = (k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$$

Step 4: Now re-arrange the expression for $n = k+1$ *to look like the answer for* $n = k$ *plus any extra terms. These terms should have the required divisor as a common factor:*

$$k^3 + 3k^2 + 3k + 1 + 2k + 2 = \boldsymbol{k^3} + \boldsymbol{2k} + 3k^2 + 3k + 3$$

$$(k+1)^3 + 2(k+1) = 3m + 3k^2 + 3k + 3 = 3(m + k^2 + k + 1)$$

✓ *True*

Step 5: Write conclusion:

If true for $n = k$ then true for $n = k+1$ and since true for $n = 1$, then by induction, it is true $\forall n \in \mathbb{N}$.

Reminder: For divisible by 6, show divisible by 2 and divisible by 3 thus . .

❹ Prove by induction that $9^n + 7$ is divisible by 8, $\forall n \in \mathbb{N}$.

Step 1: Let $n = 1$*:* $9^1 + 7 = 9 + 7 = 16, \quad 8|16$ ✓ *True*

Step 2: Assume true for $n = k$*:* $8|9^n + 7 \Rightarrow 9^n + 7 = 8m, m \in \mathbb{N}$

Step 3: Get expression for $k + 1$*:* $9^{(k+1)} + 7$

Step 4: Now re-arrange the expression to look like the answer for $n = k$ *using the laws of indices:*

$$9^{(k+1)} + 7 = 9^k \times 9 + 7 = 9^k \times (8 + 1) + 7$$

Multiply out the bracket: $8 \times 9^k + 1 \times 9^k + 7 = 8 \times 9^k + \mathbf{9^k + 7}$

$$9^{(k+1)} + 7 = 8 \times 9^k + \mathbf{8m} = 8(9^k + m)$$

✓ *True*

Step 5: Write conclusion:

If true for $n = k$ then true for $n = k + 1$ and since true for $n = 1$, then by induction, it is true $\forall n \in \mathbb{N}$.

Prove by induction that*:*

1. $3|n^3 - n$
2. $6|n^3 - n$
3. $10|6^n + 4$
4. $4|5^n + 3$
6. $64|9^n - 8n - 1$
7. $5|8^n + 3^{n-2}$

Left Field:

- DeMoivre (2012 Q16)
- Binomial (2015 Q9)
- Matrices (2006 Q13)
- n^{th} Derivative (2004 Q12)
- Greater or Less Than (2007 Q12)

Bk 3 P141 Ex3A
Q1, 3, 5, 7

Bk 3 P141 Ex3B
Q1

DeMoivre:

❺ Prove by induction that if $z = r(\cos\theta + i\sin\theta)$ then $z^n = r^n(\cos n\theta + i\sin n\theta)$

Step 1: Let $n = 1$: $z^1 = r^1(\cos 1\theta + i\sin 1\theta) = r(\cos\theta + i\sin\theta)$ ✓ *True*

Step 2: Assume true for $n = k$: $z^k = r^k(\cos k\theta + i\sin k\theta)$

Step 3: Get expression for $k+1$: $z^{k+1} = r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta)$

Step 4: Using the laws of indices and compound angle formulae:

$$z^{k+1} = z^k \times z^1$$

$$z^{k+1} = r^k(\cos k\theta + i\sin k\theta) + r(\cos\theta + i\sin\theta)$$

$$z^{k+1} = r^{k+1}(\cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta)$$

$$z^{k+1} = r^{k+1}(\cos k\theta\cos\theta - \sin k\theta\sin\theta + i\sin k\theta\cos\theta + i\cos k\theta\sin\theta)$$

$$z^{k+1} = r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta)$$

Step 5: Write conclusion:

If true for $n = k$ then true for $n = k+1$ and since true for $n = 1$, then by induction, it is true $\forall n \in \mathbb{N}$.

Binomial Theorem:

❻ Prove by induction that $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$

Step 1: Let $n = 1$: $(x+y)^1 = \sum_{r=0}^{n} \binom{1}{r} x^{1-r} y^r = \binom{1}{0} x^{1-0} y^0 + \binom{1}{1} x^{1-1} y^1 = x + y$

$(x+y)^1 = x + y$ ✓ *True*

Step 2: Assume true for $n = k$: $(x+y)^k = \sum_{r=0}^{n} \binom{k}{r} x^{k-r} y^r$

Step 3: Get expression for $k+1$: $(x+y)^{k+1} = \sum_{r=0}^{n} \binom{k+1}{r} x^{k+1-r} y^r$

Step 4: Using the laws of indices:

$$(x+y)^{k+1} = (x+y)^k \times (x+y)^1$$

$$(x+y)^{k+1} = (x+y) \sum_{r=0}^{n} \binom{k}{r} x^{k-r} y^r$$

$$(x+y)^{k+1} = x \sum_{r=0}^{n} \binom{k}{r} x^{k-r} y^r + y \sum_{r=0}^{n} \binom{k}{r} x^{k-r} y^r$$

$$= x\binom{k}{0} x^k y^0 + x\binom{k}{1} x^{k-1} y^1 + x\binom{k}{2} x^{k-2} y^2 + \cdots + y\binom{k}{0} x^k y^0 + y\binom{k}{1} x^{k-1} y^1 + y\binom{k}{2} x^{k-2} y^2 + \cdots$$

$$= \binom{k}{0} x^{k+1} y^0 + \binom{k}{1} x^k y^1 + \binom{k}{2} x^{k-1} y^2 + \cdots + \binom{k}{0} x^{k+1} y^0 + \binom{k}{1} x^k y^1 + \binom{k}{2} x^{k-1} y^2 + \cdots$$

$$= \binom{k}{0} x^{k+1} + \left[\binom{k}{0} + \binom{k}{1}\right] x^k y + \left[\binom{k}{1} + \binom{k}{2}\right] x^{k-1} y^2 + \cdots + \binom{k}{k} y^{k+1}$$

From $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ *we get that* $\binom{k}{0} + \binom{k}{1} = \binom{k+1}{1}$, $\binom{k}{1} + \binom{k}{2} = \binom{k+1}{2}$, *etc*

So:

$$\binom{k}{0} x^{k+1} + \left[\binom{k}{0} + \binom{k}{1}\right] x^k y + \left[\binom{k}{1} + \binom{k}{2}\right] x^{k-1} y^2 + \cdots + \binom{k}{k} y^{k+1}$$

$$= \binom{k}{0} x^{k+1} + \binom{k+1}{1} x^k y + \binom{k+1}{2} x^{k-1} y^2 + \cdots + \binom{k}{k} y^{k+1}$$

From $\binom{k}{0} = \binom{k+1}{0} = 1$ *and* $\binom{k}{k} = \binom{k+1}{k+1} = 1$

$$\binom{k}{0} x^{k+1} + \binom{k+1}{1} x^k y + \binom{k+1}{2} x^{k-1} y^2 + \cdots + \binom{k}{k} y^{k+1}$$

$$= \binom{k+1}{0} x^{k+1} + \binom{k+1}{1} x^k y + \binom{k+1}{2} x^{k-1} y^2 + \cdots + \binom{k+1}{k+1} y^{k+1}$$

$$= \sum_{r=0}^{n} \binom{k+1}{r} x^{k+1-r} y^r$$

✓ *True*

Step 5: Write conclusion:

If true for $n = k$ then true for $n = k+1$ and since true for $n = 1$, then by induction, it is true $\forall n \in \mathbb{N}$.

❼ Prove by induction that $\binom{n+2}{3} - \binom{n}{3} = n^2$ for integers $n \geq 3$

Let $n = 3$*:* $\binom{3+2}{3} - \binom{3}{3} = \frac{(3+2)!}{(3+2-3)!3!} - \frac{3!}{(3-3)!3!} = \frac{120}{12} - \frac{6}{6} = 9 = 3^2$

✓ *True*

Assume true for $n = k$*:* $\binom{k+2}{3} - \binom{k}{3} = k^2$

Expression for $k+1$*:* $\binom{k+1+2}{3} - \binom{k+1}{3} = \binom{k+3}{3} - \binom{k+1}{3} = (k+1)^2$

From $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ *we get* $\binom{k+3}{3} = \binom{k+2}{2} + \binom{k+2}{3}$

and $\binom{k+1}{3} = \binom{k}{2} + \binom{k}{3}$

So $\binom{k+3}{3} - \binom{k+1}{3} = \binom{\boldsymbol{k+2}}{\boldsymbol{2}} + \binom{k+2}{3} - \binom{k}{2} - \binom{\boldsymbol{k}}{\boldsymbol{3}} = k^2 + \binom{k+2}{2} - \binom{k}{2}$

$$\binom{k+2}{2} = \frac{(k+2)!}{(k+2-2)!\,2!} = \frac{(k+2)(k+1)k\ldots}{k(k-1)\ldots 2} = \frac{(k+2)(k+1)}{2}$$

$$\binom{k}{2} = \frac{(k)!}{(k-2)!\,2!} = \frac{k(k-1)(k-2)\ldots}{(k-2)(k-1)\ldots 2} = \frac{k(k-1)}{2}$$

$$\binom{k+2}{2} - \binom{k}{2} = \frac{(k+2)(k+1)}{2} - \frac{k(k-1)}{2} = \frac{k^2+3k+2-k^2+k}{2} = 2k+1$$

$k^2 + \binom{k+2}{2} - \binom{k}{2} = k^2 + 2k + 1 = (k+1)^2$ ✓ *True*

If true for $n = k$ then true for $n = k+1$ and since true for $n = 1$, then by induction, it is true $\forall n \in \mathbb{N}$.

Matrices:

❽ For $A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$, use induction to prove that $A^n = \begin{pmatrix} 1 & 0 \\ 1-2^n & 2^n \end{pmatrix}$ for all positive integers.

Let $n = 1$*:* $A^1 = \begin{pmatrix} 1 & 0 \\ 1-2^1 & 2^1 \end{pmatrix} = A^n = \begin{pmatrix} 1 & 0 \\ 1-2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$

✓ *True*

Assume true for $n = k$*:* $A^k = \begin{pmatrix} 1 & 0 \\ 1-2^k & 2^k \end{pmatrix}$

Expression for $k+1$*:* $A^{k+1} = \begin{pmatrix} 1 & 0 \\ 1-2^{k+1} & 2^{k+1} \end{pmatrix}$

Using the laws of indices: $A^{k+1} = A^k A^1$

$$A^k A^1 = \begin{pmatrix} 1 & 0 \\ 1-2^k & 2^k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 + 0 \times -1 & 0 + 0 \times 2 \\ \mathbf{(1-2^k) \times 1 - 1 \times 2^k} & 0 + 2^k \times 2 \end{pmatrix}$$

NB - $\mathbf{(1-2^k) \times 1 - 1 \times 2^k} = 1 - 2^k - 2^k = 1 - 2 \times 2^k = 1 - 2^{k+1}$

So $A^k A^1 = \begin{pmatrix} 1 & 0 \\ 1-2^{k+1} & 2^{k+1} \end{pmatrix}$

✓ *True*

If true for $n = k$ then true for $n = k+1$ and since true for $n = 1$, then by induction, it is true $\forall n \in \mathbb{N}$.

Smaller or Larger than:

❾ Prove by induction that $2^n > n$, $n \geq 1$, $n \in \mathbb{N}$.

Let $n = 1$*:* $2^1 = 2$ *which is greater than* 1

✓ *True*

Assume true for $n = k$*:* $2^k > k$

Expression for $k+1$*:* $2^{k+1} > k + 1$

Using the laws of indices: $2^{k+1} = 2^k \times 2 \Rightarrow 2^{k+1} > 2k$

Since $2k > k + 1$ *as* $k > 1$ *and* $2^{k+1} > 2k$ *then* $2^{k+1} > k + 1$

If true for $n = k$ then . . .