Unit 2: Applications in Algebra and Calculus (H7X1 77)

Applying algebraic and calculus skills to problems in context

Area and Volume

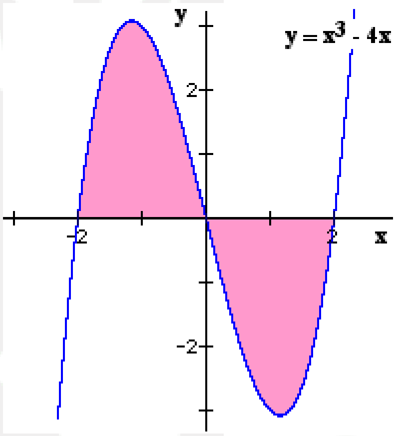
In Higher Maths we learned how to calculate the area between a curve and the x-axis (with upper and lower limit) as well as the area between two curves:

❶ Find the area enclosed by the curve $y = x^3 - 4x$, the x-axis and the lines $x = -2$ and $x = 2$.

As we can see, the curve appears both above and below the x-axis so we need to treat the two areas separately. The function is odd so:

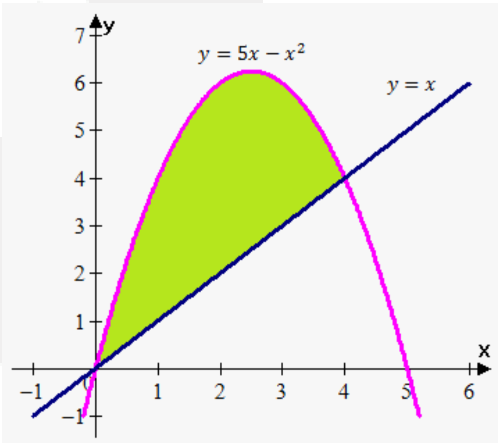
$$A = 2\int_{-2}^{0}(x^3 - 4x)dx = 2\left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0}$$

$$A = 2\left[(0) - \left(\frac{16}{4} - 8\right)\right] = 2(4) = 8\,units^2$$



❷ Find the area enclosed by the curves $y = 5x - x^2$ and $y = x$.

Points of intersection at:
$x = 5x - x^2 \Rightarrow x^2 - 4x = 0 \Rightarrow x(x-4) = 0$



$$A = \int_0^4 (5x - x^2)dx - \int_0^4 (x)dx$$

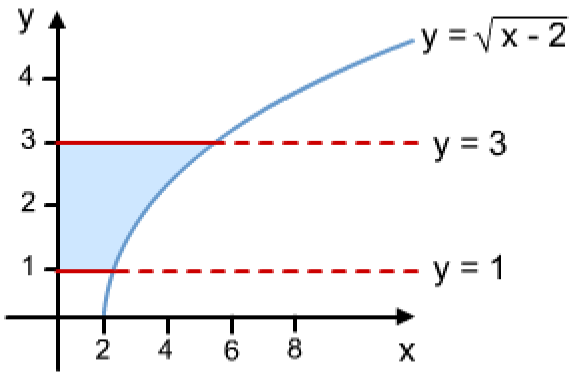
$$A = \int_0^4 (4x - x^2)dx = \left[2x^2 - \frac{x^3}{3}\right]_0^4$$

$$A = \left(2(4)^2 - \frac{4^3}{3}\right) - 0 = 32 - \frac{64}{3} = \frac{32}{3}\ units^2$$

Area between a curve and the y-axis

The method is the same but we must make sure that the function is expressed as $f(y)$ i.e. **make x the subject** - $f^{-1}(x)$.

❸ Find the area enclosed by the y-axis, the lines $= 1$, $y = 3$ and the curve $y = \sqrt{x-2}$

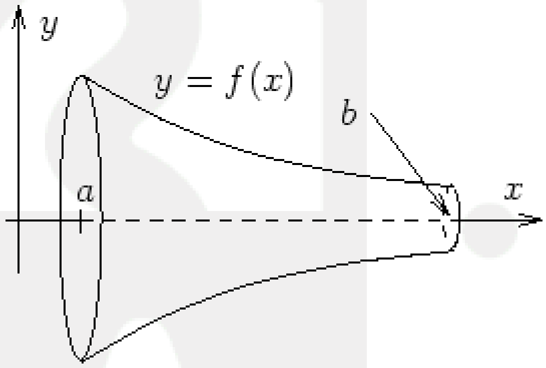


$$y = \sqrt{x-2} \Rightarrow x = y^2 + 2$$

$$A = \int_1^3 (y^2+2)dy = \left[\frac{y^3}{3} + 2y\right]_1^3$$

$$A = \left[\left(\frac{3^3}{3} + 6\right) - \left(\frac{1^3}{3} + 2\right)\right] = 15 - 2\frac{1}{3} = 12\frac{2}{3}\ units^2$$

Volumes of Revolution



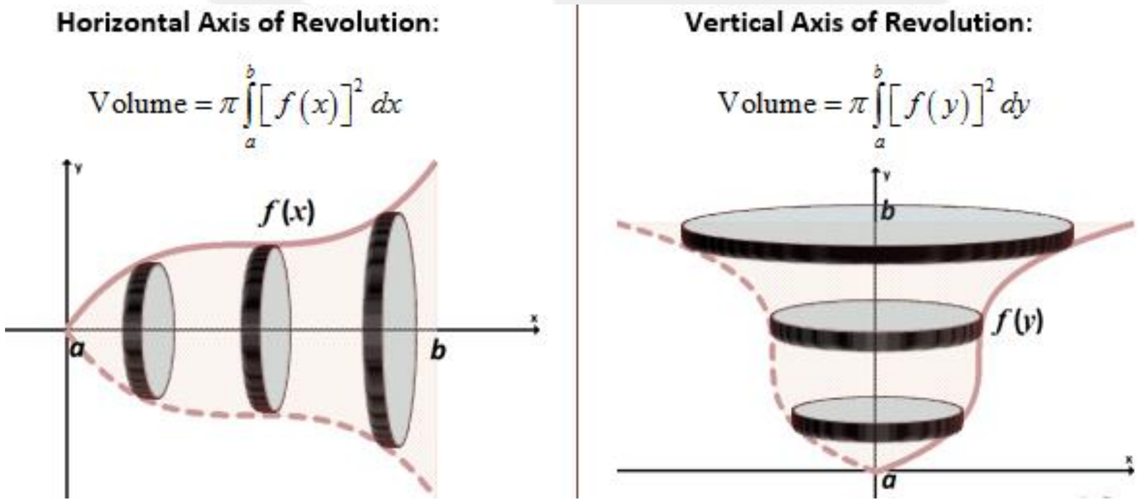
If we rotate a curve about the x-axis:

$$V = \pi \int_a^b (f(x))^2 dx$$

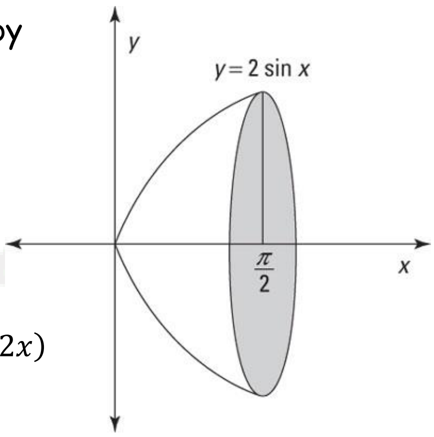
If we rotate a curve about the y-axis:

$$V = \pi \int_{f(a)}^{f(b)} (f(y))^2 dy$$

This method is known as the DISC method as it has been arrived at by slicing the area up into circular strips:



❹ Find the volume of revolution obtained by rotating the curve $y = 2\sin x$ between $x = 0$ and $x = \frac{\pi}{2}$ about the x-axis.



$$V = \pi \int_0^{\frac{\pi}{2}} (2\sin x)^2 dx = 4\pi \int_0^{\frac{\pi}{2}} (\sin^2 x) dx$$

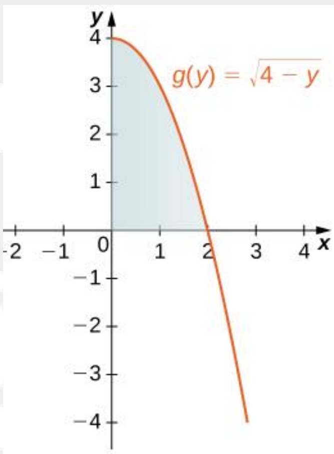
$$\cos 2x = 1 - 2\sin^2 x \Rightarrow \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

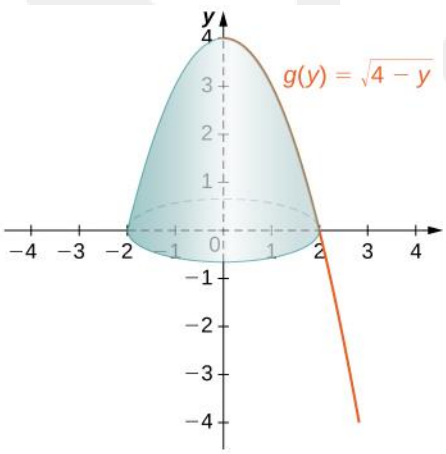
$$V = 2\pi \int_0^{\frac{\pi}{2}} (1 - \cos 2x) dx = 2\pi \left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}} = 2\pi\left[\left(\frac{\pi}{2} - 0\right) - (0)\right] = \pi^2\ units^3$$

❺ Find the volume of revolution obtained by rotating the curve $g(x) = 4 - x^2$ between $x = 0$ and $x = 2$ about the y-axis.

Find new limits: $x = 0 \Rightarrow g(0) = 4$ *and* $x = 2 \Rightarrow g(2) = 0$

Make x the subject: $y = 4 - x^2 \Rightarrow x^2 = 4 - y \Rightarrow x = \sqrt{4 - y}$

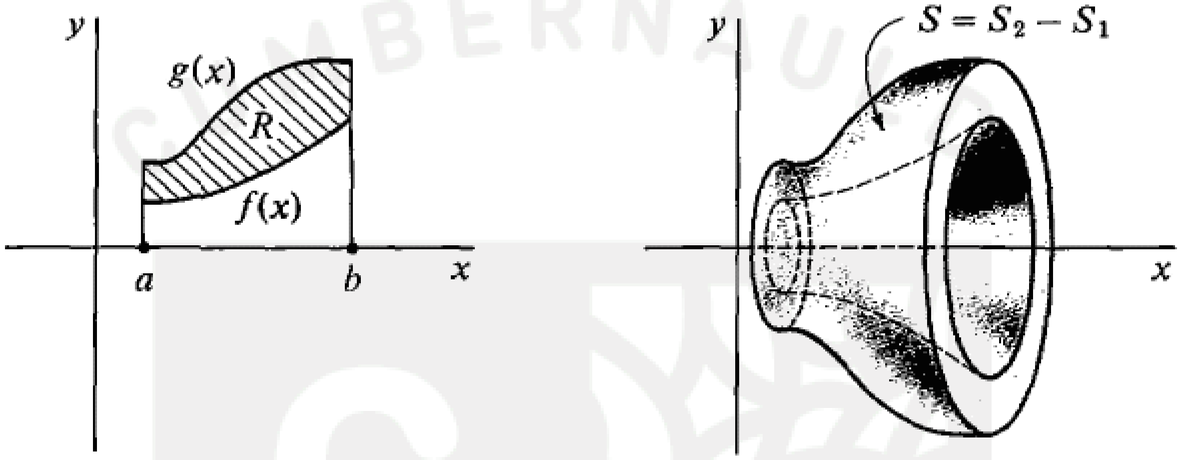




$$V = \pi \int_{g(a)}^{g(b)} (g(y))^2 dy = \pi \int_0^4 \left(\sqrt{4 - y}\right)^2 dy = \pi \int_0^4 (4 - y) dy$$

$$V = \pi \left[4y - \frac{y^2}{2}\right]_0^4 = \pi[(16 - 8) - 0] = 8\pi\ units^3$$

NB – To find the volume of a solid involving two functions and the area enclosed, we have to find the volumes first and then subtract.



$$V = \int_a^b \pi(g(x))^2\,dx - \int_a^b \pi(f(x))^2\,dx.$$

<u>Practice Questions:</u>

Bk1 P88
Ex10A
Odd Numbers

1. Find the area enclosed by the curve $y = x + \frac{1}{x}$, the lines $x = 4$, $x = 6$ and the x-axis.
2. Find the area between the curve $y = 5x - x^2$ and the line $y = x + 3$.
3. Find the volume of the solid formed by rotating about the x-axis the region bounded by the curve $y = \frac{\sqrt{x}+1}{\sqrt{x}}$, the x-axis and the lines $x = 1$ and $x = 4$.
4. Find the volume generated when the region bound by the curves $y = x^2$ and $y^2 = x$ is rotated through one revolution about the y-axis.
5. When the area bounded by the curve $y = 15x(k - x)$ and the x-axis is rotated through one revolution about the x-axis, the volume generated is 240π. Find the value of k.

$$\left\{10 + \ln\frac{3}{2},\ 1\frac{1}{3},\ \pi(7 + \ln 4),\ \frac{3\pi}{10},\ k = 2\right\}$$

Rate of Change

Questions involving "Rate of Change" almost always involve time as a variable combined with **Parametric Differentiation**. Implicit Differentiation and Rectilinear Motion including Separating Variables and have also appeared.

Rectilinear Motion (Reminder):

Displacement from the origin is a function of time and is denoted:

$$x = f(t) \qquad \text{or} \qquad s = f(t)$$

Velocity is given by: $v = \frac{dx}{dt}$

Acceleration is given by: $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

Conversely: $v = \int a\,dt$ and $x = \int v\,dt$ or $s = \int v\,dt$

❻ A body starts from rest and its velocity is given by $v = 3t^2 + 4t - 1$. Find its displacement, velocity and acceleration when $t = 4$.

Velocity: $v = 3 \times 4^2 + 4 \times 4 - 1 = 63\,ms^{-1}$

Displacement: $x = \int (3t^2 + 4t - 1)\,dt = t^3 + 2t^2 - t + c$

$x = 0$ *when* $t = 0$ *(starts from rest) so* $c = 0$

$x = 4^3 + 2 \times 4^2 - 4 = 92\,m$

Acceleration: $a = \frac{dv}{dt} = 6t + 4 \Rightarrow 6 \times 4 + 4 = 28\,ms^{-2}$

Derivative of Parametric Equations reminder:

We use the Chain Rule to differentiate Parametric Equations:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \quad \text{where} \quad \frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} \quad \text{i.e. invert } \frac{dx}{dt} \text{ to get } \frac{dt}{dx}$$

Implicit Differentiation reminder:

- *Differentiate both sides with respect to* x
- *The derivative of* y *is* $\frac{dy}{dx}$
- *Use the chain rule for* $\frac{d}{dx}(y^2)$*:* $\frac{d}{dx}(y)^2 = 2y \times \frac{dy}{dx}$
- *use the product rule for* $\frac{d}{dx}(xy)$

One Variable – Parametric differentiation.

Formulae that have only one variable such as the volume of cubes and spheres or surface area of cubes and spheres will normally require parametric differentiation – sometimes more than once.

❼ The height of a cube is increasing at the rate of 2 cms^{-1}.
Find the rate of increase of the volume when the height of the cube is 8 cm.

The "rate of increase of the volume" is $\frac{dV}{dt}$

We know that the volume of a cube is found using $V = l^3$

By parametric differentiation: $\frac{dV}{dt} = \frac{dV}{dl} \times \frac{dl}{dt} = 3l^2 \times 2 = 6l^2$

When $l = 8$ *then* $\frac{dV}{dt} = 6l^2 = 6 \times 8^2 = 384cm^3s^{-1}$

❽ A spherical balloon is being inflated. When the radius is 12 cm, the surface area is increasing at a rate of $240\pi cm^2 s^{-1}$. Find the rate at which the volume is increasing at this moment.

The "rate at which the volume is increasing" is $\frac{dV}{dt}$

We would normally be reminded that $V = \frac{4}{3}\pi r^3$ *and* $SA = 4\pi r^2$

By parametric differentiation: $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ *but we don't have* $\frac{dr}{dt}$

We do have $\frac{dA}{dt} = 240\pi$ *and can use this to find* $\frac{dr}{dt}$

By parametric differentiation: $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 8\pi r \times \frac{dr}{dt} = 240\pi$

We know that $r = 12$ *so* $\frac{dr}{dt} = \frac{240\pi}{96\pi} = 2.5$

Finally: $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times 2.5 = 4\pi(12)^2 \times 2.5 = 1440\pi cm^3s^{-1}$

Two Variables – Parametric with Implicit differentiation.

Formulae that have two variables such as the volume and surface area of cylinders will normally require implicit differentiation.

❾ The radius of a cylindrical balloon is decreasing at the rate of 0.1 ms^{-1}, while the height is increasing at the rate of 0.02 ms^{-1}.

Find the rate of change of the volume when the radius is 0.5 metres and the height is 3 metres.

The "rate of change of the volume" is $\frac{dV}{dt}$

We would normally be given that $V = \pi r^2 h$

By parametric differentiation: $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

We know that $\frac{dr}{dt} = -0.1$ *and that* $\frac{dh}{dt} = 0.02$

Using implicit differentiation combined with the product rule:

$$u = \pi r^2 \text{ and } v = h \Rightarrow \frac{dv}{dr} = u'v + uv' = 2\pi rh + \pi r^2 \frac{dh}{dr}$$

Parametric again: $\frac{dh}{dr} = \frac{dh}{dt} \times \frac{dt}{dr} = 0.02 \times \frac{1}{-0.1} = -0.2$

So: $\frac{dv}{dr} = 2\pi rh + \pi r^2 \frac{dh}{dr} = 2\pi \times 0.5 \times 3 + \pi \times 0.5^2 \times (-0.2) = 2.95\pi$

Finally: $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 2.95\pi \times -0.1 = -0.295\pi\ m^3 s^{-1}$

Bk2 P53/54 Ex2A
Q4-6

P54-56 Ex2B
Q5, 6 & 10

Optimisation

We met these types of problems in Higher Maths. The following is merely an extension of that work:

Hints for solving these types of problems:

1. Read the question carefully, understanding what is required.
2. Draw a sketch if appropriate.
3. Convert the information into a formula or formulae.
4. Identify what variable is to be optimised and express the function in this variable only.
5. Find the Stationary Points/Critical Points/Extrema of the function.
6. Answer the question.

❶⓿ The fuel efficiency, F, in km per litre, of a vehicle varies with its speed s km per hour, and for a particular vehicle the relationship is thought to be

$$F = 15 + e^x(\sin x - \cos x - \sqrt{2}) \text{ where } \quad x = \frac{\pi(s-40)}{80}$$

For speeds in the range $40 \leq s \leq 120$ km per hour.

What is the greatest ans least efficiency over the range and what speeds do they occur?

$$\frac{dF}{dx} = 0 + e^x(\cos x + \sin x) + e^x(\sin x - \cos x - \sqrt{2}) = e^x(2\sin x - \sqrt{2})$$

SPs at $\frac{dF}{dx} = 0$ *so* $e^x(2\sin x - \sqrt{2}) = 0 \Rightarrow 2\sin x = \sqrt{2} \Rightarrow \sin x = \frac{1}{\sqrt{2}}$

$x = \frac{\pi}{4}$ *and* $x = \frac{3\pi}{4}$ *gives* $F \approx 11.9$ *and* $\underline{F \approx 15}$

At the end points: $40 \leq s \leq 120$ *and* $x = \frac{\pi(s-40)}{80}$ *so* $0 \leq x \leq \pi$

$x = 0$ *and* $x = \pi$ *gives* $F \approx 12$ *and* $\underline{F \approx 5.4}$

Greatest efficiency is 15 km/litre $\left(\frac{3\pi}{4} = \frac{\pi(s-40)}{80}\right) \Rightarrow s = 100$ *km/hr*

Least efficiency is 5·4 km/litre $\left(\pi = \frac{\pi(s-40)}{80}\right) \Rightarrow s = 120$ *km/hr*

Bk1 P63 Ex4A
Odd Numbers