Unit 3: Geometry, Proof and Systems of Equations (H7X3 77) - Vectors

REMINDERS

If P is the point (x, y, z) then the POSITION vector $\underline{p} = \overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\overrightarrow{AB} = \underline{b} - \underline{a}$ mid-point of $AB = \frac{1}{2}(\underline{a} + \underline{b})$ $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$

$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ For $\underline{v} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 2\underline{i} - 3\underline{j} + \underline{k}$

$|\underline{v}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$ $-2\underline{v} = -2\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ -2 \end{pmatrix}$

Section Formula: If P splits the line AB in the ratio $m{:}n$ then $\underline{p}$ is given by:

$$\underline{p} = \frac{m\underline{b} + n\underline{a}}{m + n}$$

❶ For $P(3,2,-1)$ and $Q(5,-3,7)$, find R given that R splits PQ in the ratio $2{:}3$

$$\underline{r} = \frac{2\underline{q} + 3\underline{p}}{5} \Rightarrow 5\underline{r} = 2\underline{q} + 3\underline{p} = 2\begin{pmatrix} 5 \\ -3 \\ 7 \end{pmatrix} + 3\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 19 \\ 0 \\ 11 \end{pmatrix} \Rightarrow \underline{r} = \begin{pmatrix} \frac{19}{5} \\ 0 \\ \frac{11}{5} \end{pmatrix} \Rightarrow R\left(\frac{19}{5}, 0, \frac{11}{5}\right)$$

Scalar Product: (Dot product) $\underline{a}.\underline{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\underline{a}||\underline{b}|\cos\theta$

❷ If $\underline{a} = 5\underline{i} + 3\underline{j} + 7\underline{k}$ and $\underline{b} = 2\underline{i} - 8\underline{j} + 4\underline{k}$, calculate the angle between the vectors.

$$\cos\theta = \frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|} = \frac{10 - 24 + 28}{\sqrt{83}\sqrt{84}} = \frac{14}{\sqrt{6972}} \Rightarrow \theta = 80.3°$$

Bk3 P44 Ex1
Q1, 4, 6, 8, 10, 13

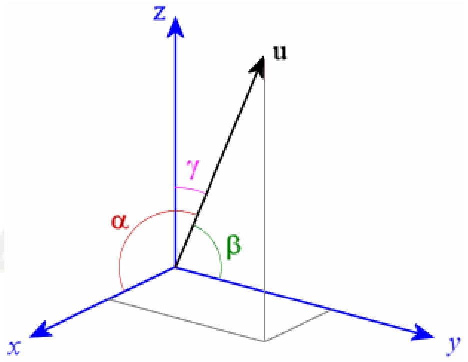
DIRECTION RATIOS and DIRECTION COSINES

If $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ then the **direction ratio** is given by $u_1{:}u_2{:}u_3$ in its simplest form.

NB – If two vectors have equal direction ratios then they are parallel OR if they share a common point then collinear.

For **unit vector** $\boldsymbol{u}$ i.e. $|\underline{u}| = 1$ and where:

α *is the angle the vector makes with the* $x-$*axis*
β *is the angle the vector makes with the* $y-$*axis*
γ *is the angle the vector makes with the* $z-$*axis*



$$u_1 = \boldsymbol{u}.\boldsymbol{i} = |\boldsymbol{u}||\boldsymbol{i}|\cos\alpha = \cos\alpha$$
$$u_2 = \boldsymbol{u}.\boldsymbol{j} = |\boldsymbol{u}||\boldsymbol{j}|\cos\beta = \cos\beta$$
$$u_3 = \boldsymbol{u}.\boldsymbol{k} = |\boldsymbol{u}||\boldsymbol{k}|\cos\gamma = \cos\gamma$$

Giving $\underline{u} = \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix}$ which are known as the **direction cosines** of $\boldsymbol{u}$

❸ Find the direction ratio and direction cosines of the vector $\boldsymbol{a} = \begin{pmatrix} 6 \\ 8 \\ 24 \end{pmatrix}$

Direction ratio: $6:8:24 = 3:4:12$

Direction cosines: $|\boldsymbol{a}| = \sqrt{6^2 + 8^2 + 24^2} = 26$

$$\boldsymbol{u_a} = \frac{1}{26}\begin{pmatrix} 6 \\ 8 \\ 24 \end{pmatrix} = \begin{pmatrix} 6/26 \\ 8/26 \\ 24/26 \end{pmatrix} = \begin{pmatrix} 3/13 \\ 4/13 \\ 12/13 \end{pmatrix}$$ so $\cos\alpha = \frac{3}{13}$, $\cos\beta = \frac{4}{13}$ and $\cos\gamma = \frac{12}{13}$

NB - $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

❹ Find a unit vector parallel to $\boldsymbol{p} = 3\boldsymbol{i} - 4\boldsymbol{j} + 12\boldsymbol{k}$

$|\boldsymbol{p}| = \sqrt{3^2 + (-4)^2 + 12^2} = 13$ so $\boldsymbol{u_p} = \frac{1}{13}\begin{pmatrix} 3 \\ -4 \\ 12 \end{pmatrix} = \begin{pmatrix} 3/13 \\ -4/13 \\ 12/13 \end{pmatrix}$

❺ Use direction ratios to prove that $A(0,-3,-1)$, $B(6,-6,5)$ and $C(4,-5,3)$ are collinear.

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 6 \\ -6 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 6 \end{pmatrix}$ so direction ratio = $2:-1:2$

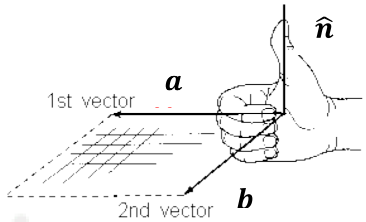
$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} = \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix} - \begin{pmatrix} 6 \\ -6 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$ so direction ratio = $2:-1:2$

$\overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear since they have the same direction ratios and B is a common point.

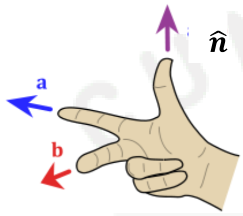
NB - C divides AB in the ratio $-3:1$

Bk3 P46 Ex2
Odd numbers

<u>VECTOR PRODUCT</u> (Cross Product) $\boldsymbol{a} \times \boldsymbol{b}$ –" $\boldsymbol{a}$ cross $\boldsymbol{b}$"



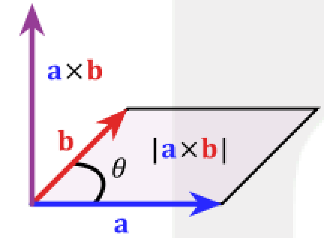
Two non-parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ define a plane:
$\hat{\boldsymbol{n}}$ is a unit vector perpendicular to this plane so that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\hat{\boldsymbol{n}}$ form a <u>Right Hand system</u> of vectors:



$\boldsymbol{a}$ is the pointing finger

$\boldsymbol{b}$ is the middle finger

$\hat{\boldsymbol{n}}$ is the thumb (known as the <u>**normal**</u>)



<u>$\boldsymbol{a} \times \boldsymbol{b}$ is a vector with the same sense and direction as $\hat{\boldsymbol{n}}$</u>

$|\boldsymbol{a} \times \boldsymbol{b}|$ is the area of the parallelogram defined by $\boldsymbol{a}$ and $\boldsymbol{b}$

NB – as with scalar product, <u>**vectors must be tail-to-tail**</u>

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta$$

NB: - $|\boldsymbol{a} \times \boldsymbol{a}| = |\boldsymbol{a}||\boldsymbol{a}| \sin 0 = 0$
- Parallel vectors have vector product 0

From the formula sheet:

Vector product

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta\,\hat{\boldsymbol{n}}$$

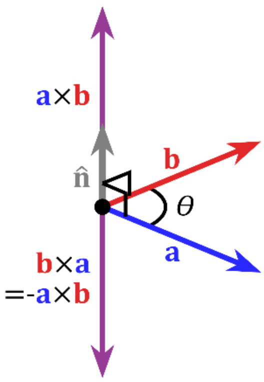
Component form: $$\boldsymbol{a} \times \boldsymbol{b} = (a_2b_3 - a_3b_2)\boldsymbol{i} - (a_1b_3 - a_3b_1)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}$$

where $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

❻ For $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ find $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{b} \times \boldsymbol{a}$

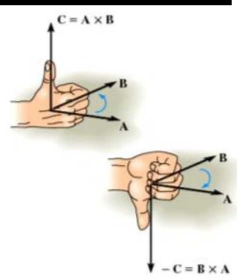
$$\boldsymbol{a} \times \boldsymbol{b} = (2 - -3)\boldsymbol{i} - (1 - 6)\boldsymbol{j} + (-1 - 4)\boldsymbol{k} = 5\boldsymbol{i} - 5\boldsymbol{j} - 5\boldsymbol{k}$$

$$\boldsymbol{b} \times \boldsymbol{a} = (-3 - 2)\boldsymbol{i} - (6 - 1)\boldsymbol{j} + (4 - -1)\boldsymbol{k} = -5\boldsymbol{i} + 5\boldsymbol{j} + 5\boldsymbol{k}$$



As you can hopefully see: $\boldsymbol{b} \times \boldsymbol{a} = -(\boldsymbol{a} \times \boldsymbol{b})$

Still a Right Hand system but thumb pointing down.



Bk3 P49 Ex3A
Q2, 3a, 3b

Some useful results to learn:

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$$

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k} \qquad \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i} \qquad \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$$

As with the Scalar product formulae and finding the angle between two vectors, the two formulae (area and cross product) permit us to find $\hat{\boldsymbol{n}}$.

❼ Find a unit vector perpendicular to both $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ & $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

$\hat{\boldsymbol{n}}$ is a unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$. Re-arranging the cross product formulae, can you see that:

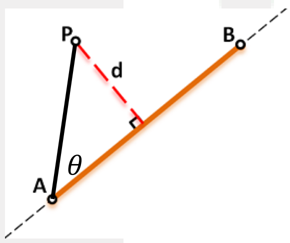
$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{a} \times \boldsymbol{b}}{|\boldsymbol{a} \times \boldsymbol{b}|}$$

$$\boldsymbol{a} \times \boldsymbol{b} = (2 - 1)\boldsymbol{i} - (4 - -1)\boldsymbol{j} + (-2 - 1)\boldsymbol{k} = \boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}$$

$$|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{1^2 + 5^2 + (-3)^2} = \pm\sqrt{35}$$

$$\hat{\boldsymbol{n}} = \pm\frac{1}{\sqrt{35}}(\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k})$$

❽ Calculate the shortest distance from the point $P(1, 2, 3)$ to the line passing through $A(1, 3, -2)$ and $B(2, 2, -1)$.



Shortest distance is perpendicular distance d

$$\sin\theta = \frac{|d|}{|AP|} \Rightarrow |d| = |AP|\sin\theta$$

Multiplying rhs by $\frac{|AB|}{|AB|}$ *we get* $\quad |d| = \frac{|AB||AP|\sin\theta}{|AB|} = \frac{|AB \times AP|}{|AB|}$

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \overrightarrow{AP} = \boldsymbol{p} - \boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix}$$

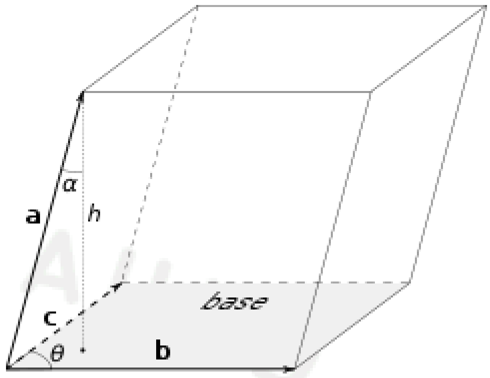
$$|\overrightarrow{AB}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3} \qquad AB \times AP = -\boldsymbol{4i} - 5\boldsymbol{j} - \boldsymbol{k}$$

$$|AB \times AP| = \sqrt{(-4)^2 + (-5)^2 + (-1)^2} = \sqrt{42}$$

$$|d| = \frac{\sqrt{42}}{\sqrt{3}} = \sqrt{14}$$ *So shortest distance =* $\sqrt{14}$ *units*

Bk3 P52 Ex4
Q2a, 2c, 3a, 3b, 8, 9a

SCALAR TRIPLE PRODUCT



A parallelepiped is formed by a set of 3 parallel planes (3D parallelogram).

It's volume can be calculated by multiplying the area of it's base and the perpendicular height $V = Ah$

The area of the base can be found using $A = |\boldsymbol{b} \times \boldsymbol{c}|$

The perpendicular height can be found using $h = |\boldsymbol{a}| \cos \alpha$

Thus $V = |\boldsymbol{b} \times \boldsymbol{c}||\boldsymbol{a}| \cos \alpha = \boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c})$

Similar result using any of the 3 planes: $V = \boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}.(\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c}.(\boldsymbol{a} \times \boldsymbol{b})$

$\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c})$ is called the **scalar triple product** and is often denoted: $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$

For $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \Rightarrow \boldsymbol{b} \times \boldsymbol{c} = \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix}$

$$\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.\begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix} = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

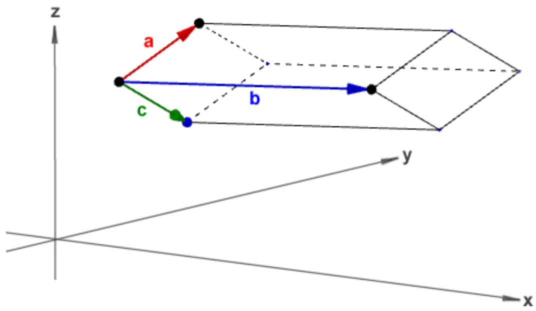
Re-arranging the middle term:

$$\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Can you see the similarity with the determinant?

$$\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

❾ Calculate the volume of the parallelepiped with $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = 4\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{c} = 2\boldsymbol{j} - \boldsymbol{k}$



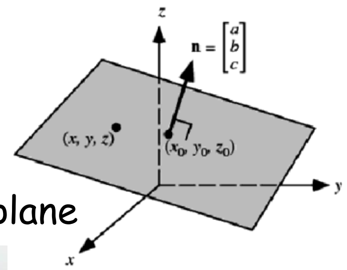
$$\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 0 & 2 & -1 \end{vmatrix} = 1(-2-0) - 1(-4-0) + 1(8-0) = -2 + 4 + 8 = 10$$

Volume of the parallelepiped $= 10\ units^3$

Bk3 P55 Ex5
Q2, 3a, 4

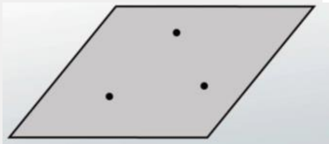
Equation of a Plane

The general equation of a plane is: $ax + by + cz = k$ where $\boldsymbol{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normal to the plane and $k = \boldsymbol{n}.\boldsymbol{p}$ where P is a point on the plane.

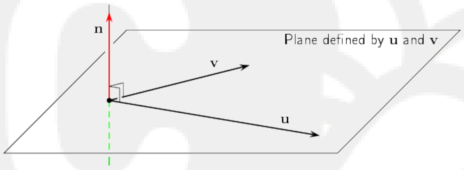


A plane can be defined in three ways:

1. By one point on the plane and a normal to the plane
2. By 3 points on the plane



3. By 2 lines



10 Find the equation of the plane perpendicular to PQ, $P(1,2,3)$ and $Q(2,1,-4)$ which contains P

$$\overrightarrow{PQ} = \boldsymbol{q} - \boldsymbol{p} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -7 \end{pmatrix}$$ This is the normal to the plane

$$\overrightarrow{PQ}.\boldsymbol{p} = \begin{pmatrix} 1 \\ -1 \\ -7 \end{pmatrix}.\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 - 2 - 21 = -22 \quad \text{so} \quad x - y - 7z = -22$$

11 Find the equation of the plane passing through the points $P(-2,1,2)$ $Q(0,2,5)$ and $R(2,-1,3)$.

We can use $\underline{PQ \times PR \ \ with\ P}$ *or* $\underline{QP \times QR \ \ with\ Q}$ *or* $\underline{RP \times RQ \ \ with\ R}$

$$\overrightarrow{PQ} = \boldsymbol{q} - \boldsymbol{p} = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \qquad \overrightarrow{PR} = \boldsymbol{r} - \boldsymbol{p} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$

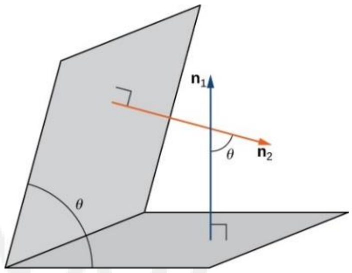
$$\boldsymbol{n} = PQ \times PR = \begin{matrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & 3 \\ 4 & -2 & 1 \end{matrix} = 7\boldsymbol{i} + 10\boldsymbol{j} - 8\boldsymbol{k}$$

Bk3 P57 Ex6
Q1c, 2c, 3a, 4a, 5a, 6a, 9

$$k = (PQ \times PR).\boldsymbol{p} = \begin{pmatrix} 7 \\ 10 \\ -8 \end{pmatrix}.\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = -20 \quad \text{so} \quad 7x + 10y - 8z = -20$$

Angle between two Planes

The angle between two planes is defined as the angle between their normals.



❶❷ Find the acute angle between the planes with equations $x + 3y - z = 5$ and $2x - y + z = -7$

Normals are: $\boldsymbol{n}_1 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

Using the dot product: $\cos\theta = \dfrac{\boldsymbol{n}_1.\boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|} = \dfrac{2-3-1}{\sqrt{11}\times\sqrt{6}} = \dfrac{-2}{\sqrt{66}}$

Bk3 P59 Ex7A
Q1 & 3

$\theta = 104.3° \Rightarrow$ acute angle $= 75.7°$

Vector Equation of a Line

We're already familiar with the idea of finding the equation of a line in 2D: the line with gradient m and a point on the line (a, b) using $y - b = m(x - a)$

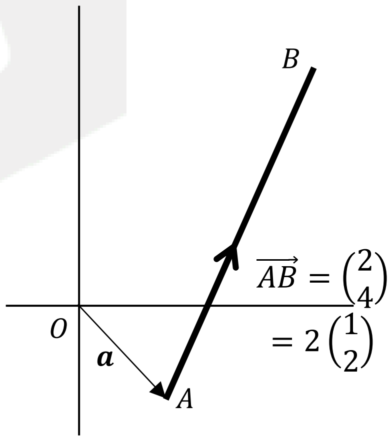
When we try to specify a line in three dimensions (or in n dimensions), however, things get more involved. It can be done without vectors, but vectors provide a really clear and quick way into the challenge.

How much information is needed in order to specify a straight line? The answer is that we need to know two things: a point through which the line passes, and the line's direction. **Both of those things can be described using vectors.**

❶❸ Finding the equation of the vector line that passes through the points A(2, -1) and B(4, 3).

Start with a position vector that takes you from the origin onto the vector line. It is usual to use the first coordinate given in the question i.e. $\boldsymbol{a}$

The direction vector of $\overrightarrow{AB}$ gives the direction of the line.



Having found a position vector and the direction vector, we have the vector equation of the line: $\boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

where $\boldsymbol{r}$ is a position vector that will take us to any point on the line depending on the value of λ

The general vector equation of a line is given as:

$$\boxed{\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{u}}$$

where R is any point on the line with position vector $\boldsymbol{r}$, $\boldsymbol{u}$ is the direction vector of the line and λ is a scalar.

❶❹ Similarly for 3D, find the equation of the vector line that passes through the points A(1, 2, 3) and B(2, 3, 5).

$$\boldsymbol{a} = \begin{pmatrix}1\\2\\3\end{pmatrix} \quad \text{and} \quad \boldsymbol{u} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix}2\\3\\5\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix}$$

$$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{u} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\2\end{pmatrix} = (\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) + \lambda(\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$$

Parametric Equation of a Line

Using $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{u}$ with $A(a_1, a_2, a_3)$ and $\boldsymbol{u} = \begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ and $R(x, y, z)$ then:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} + \lambda\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$$

Thus

$$\boxed{x = a_1 + \lambda u_1 \qquad y = a_2 + \lambda u_2 \qquad z = a_3 + \lambda u_3}$$

This is known as the PARAMETRIC EQUATION of the line

Cartesian (Symmetric) Equation of a Line

Using $\quad x = a_1 + \lambda u_1 \qquad y = a_2 + \lambda u_2 \qquad x = a_3 + \lambda u_3$

If we make λ the subject: $\lambda = \dfrac{x-a_1}{u_1}$, $\lambda = \dfrac{y-a_2}{u_2}$ and $\lambda = \dfrac{z-a_3}{u_3}$

This results in: $\boxed{\dfrac{x-a_1}{u_1} = \dfrac{y-a_2}{u_2} = \dfrac{z-a_3}{u_3}}$ which is the Cartesian Equation

From example ❶❹: $\boldsymbol{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\2\end{pmatrix}$ we get:

Parametric form: $x = \lambda + 1 \quad y = \lambda + 2 \quad z = 2\lambda + 3$ NB - $y = mx + c$ format

Cartesian form: $\dfrac{x-1}{1} = \dfrac{y-2}{1} = \dfrac{z-3}{2}$ NB – denominators = direction vector

❶❺ Find the Cartesian equation of the line through $(1, -2, 8)$ and parallel to the vector $3\boldsymbol{i} + 5\boldsymbol{j} + 11\boldsymbol{k}$

$\frac{x-1}{3} = \frac{y+2}{5} = \frac{z-8}{11}$ - *Can you spot the quick way to get this?*

❶❻ Finding the parametric equation of the line that passes through the points A(2, 1, 3) and B(3, 4, 5).

Bk3 P66 Ex9A Q1a, 2a, 3a, 4, 5, 6

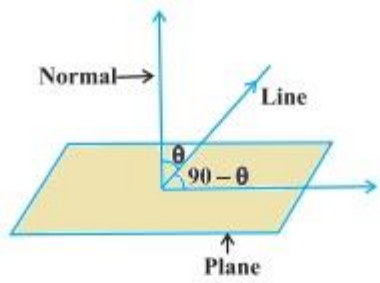
$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix}3\\4\\5\end{pmatrix} - \begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}1\\3\\2\end{pmatrix} \Rightarrow \frac{x-2}{1} = \frac{y-1}{3} = \frac{z-3}{2}$$

Bk3 P67 Ex9B Q2, 4

$x = \lambda + 2 \quad y = 3\lambda + 1 \quad z = 2\lambda + 3$

The Angle between a Line and a Plane

The angle between the normal and the line $\theta°$ can be found using the scalar product. The angle between the plane and the line is the complement of this angle i.e. $90 - \theta°$. (NB $\cos(90 - \theta°) = \sin\theta°$)



❶❼ Find the point of contact and the size of the angle between the line $\frac{x-7}{3} = \frac{y-11}{4} = \frac{z-24}{13}$ and the plane $6x + 4y - 5z = 28$

Step 1: change from Cartesian/Symmetric to Parametric:

$x = 3\lambda + 7 \quad y = 4\lambda + 11 \quad z = 13\lambda + 24$

Step 2: Substitute x, y, z into plane equation and solve for λ:

$$6(3\lambda + 7) + 4(4\lambda + 11) - 5(13\lambda + 24) = 28$$

$$-31\lambda - 34 = 28 \Rightarrow \lambda = -2$$

Step 3: Find coordinates: $x = -6 + 7 = 1 \quad y = -8 + 11 = 3 \quad z = -26 + 24 = -2$

Point of contact is $(1, 3, -2)$

Step 4: Use $\sin\theta = \cos(90 - \theta) = \frac{|\boldsymbol{a}.\boldsymbol{u}|}{|\boldsymbol{a}||\boldsymbol{b}|}$ *to find angle:*

$\boldsymbol{a} = \begin{pmatrix}6\\4\\-5\end{pmatrix}$ *and* $\boldsymbol{u} = \begin{pmatrix}3\\4\\13\end{pmatrix}$ *so* $\sin\theta = \frac{|\boldsymbol{a}.\boldsymbol{u}|}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{|18+16-65|}{\sqrt{77}\times\sqrt{194}} \Rightarrow \theta = 14.7°$

Bk3 P68 Ex10 Q1a, 1d, 1g, 2a, 3, 4a, 4c

Intersection of two Lines

❶❽ Do the lines $\frac{x-12}{5} = \frac{y+3}{-2} = \frac{z-5}{4}$ and $\frac{x-5}{1} = \frac{-y-2}{1} = \frac{z}{1}$ cross and if so, what is the point of intersection?

Step 1: Change from Cartesian/Symmetric to Parametric:

$x = 5\lambda_1 + 12$ $\quad y = -2\lambda_1 - 3$ $\quad z = 4\lambda_1 + 5$

$x = \lambda_2 + 5$ $\quad y = -\lambda_2 - 2$ $\quad z = \lambda_2$

Step 2: Equate x, y, z:

$5\lambda_1 + 12 = \lambda_2 + 5$ $\quad -2\lambda_1 - 3 = -\lambda_2 - 2$ $\quad 4\lambda_1 + 5 = \lambda_2$

Step 3: Use Simultaneous Equations with any two of these equations:

$\lambda_2 = 4\lambda_1 + 5$ from [3] so $5\lambda_1 + 12 = (4\lambda_1 + 5) + 5$ from [1] $\Rightarrow \lambda_1 = -2$

Substitute $\lambda_1 = -2$ in [3] gives $\lambda_2 = 4 \times -2 + 5 = -3$

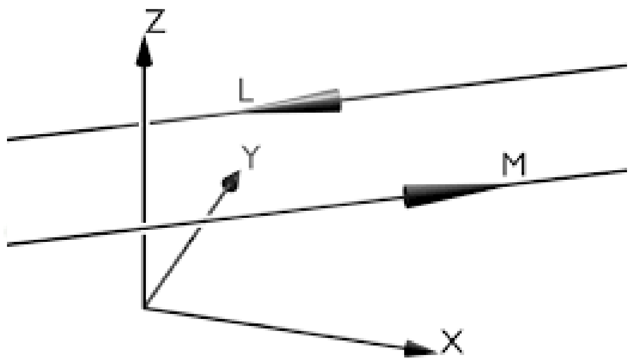
Step 4: Substitute λ_1 and λ_2 into the remaining equation, if the values satisfy this equation then there is a point of contact, if not then the lines don't cross.

$$-2\lambda_1 - 3 = -\lambda_2 - 2$$
$$-2 \times -2 - 3 = -(-3) - 2$$
$$1 = 1 \Rightarrow \textit{lines intersect}$$

Step 5: Substitute λ_1 or λ_2 into relevant parametric equation:

$x = \lambda_2 + 5 = -3 + 5 = 2$ $\quad y = -\lambda_2 - 2 = -(-3) - 2 = 5$ $\quad z = \lambda_2 = -3$

Point of contact is $(2, 1, -3)$

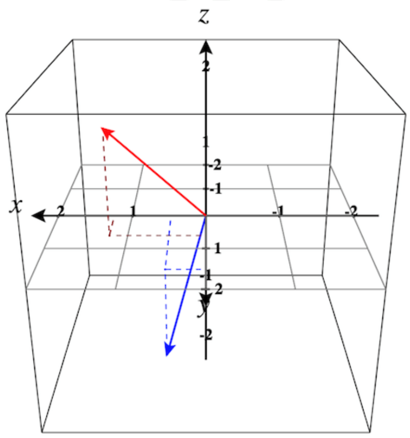


Angle between two Lines

❶❾ Find the size of the angle between the lines $\frac{x-2}{1} = \frac{y+1}{-2} = \frac{z-11}{-1}$ and $x = -\lambda_2 + 3, \quad y = \lambda_2 - 4, \quad z = -8.$

Use the dot product with the respective direction vectors:

$\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ *then converting 2nd line:*

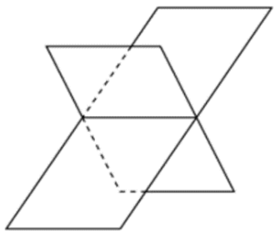


$$\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+8}{0} \Rightarrow \boldsymbol{u}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\cos\theta = \frac{-1-2+0}{\sqrt{6}\times\sqrt{2}} \Rightarrow \theta = 150°$$

Bk3 P70 Ex11
Q1, 3(use x and z)

Line of Intersection of Two Planes



Note:

- **If the planes are parallel then their normals will also be parallel.**
- **For a line of intersection, the direction vector of the line will be perpendicular to both planes (i.e. the cross product of the normals will give us the direction vector of this line)**

❷⓿ Find the equation of the line of intersection of the planes with equations $x - 2y + 3z = 1$ and $2x + y + z = -3$.

Find the vector product of the two normals (and simplify if needs be):

$$\boldsymbol{n}_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \boldsymbol{n}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \qquad \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{matrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -2 & 3 \\ 2 & 1 & 1 \end{matrix} = -5\boldsymbol{i} + 5\boldsymbol{j} + 5\boldsymbol{k} = \mathbf{1}:-\mathbf{1}:-\mathbf{1}$$

The line either crosses the x, y plane or is parallel to it. Use simultaneous equations to find a point on the line with $z = 0$ or if it is parallel then choose a similar point on the x, z plane i.e. $y = 0$

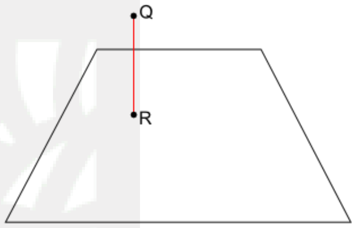
$$z = 0 \Rightarrow x - 2y = 1 \text{ and } 2x + y = -3$$

$$\Rightarrow \begin{matrix} x - 2y = 1 \\ 4x + 2y = -6 \end{matrix} \Rightarrow 5x = -5 \Rightarrow x = -1, y = -1$$

Bk3 P72 Ex12
Q1

$$(-1, -1, 0) \qquad \Rightarrow \frac{x+1}{1} = \frac{y+1}{-1} = \frac{z}{1}$$

The Shortest Distance from a Point to a Plane.



Let Q be a point lying out with the plane

RQ is the normal to the plane.

❷❶ Find the distance from the point $Q(3, 2, 1)$ and the plane with equation $x - 2y + 2z = -13$

Step 1: Find the equation of the line RQ: $\quad \boldsymbol{n}_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$$\Rightarrow \frac{x-3}{1} = \frac{y-2}{-1} = \frac{z-1}{2}$$

Step 2: Convert to parametric and substitute for x, y, z into Cartesian equation of plane to find the value of λ:

$$x = \lambda + 3 \qquad y = -\lambda + 2 \qquad z = 2\lambda + 1$$

$$(\lambda + 3) - 2(-\lambda + 2) + 2(2\lambda + 1) = \lambda + 3 + 2\lambda - 4 + 4\lambda + 2 = 7\lambda + 1 = -13$$

$$\text{so} \qquad \lambda = -2$$

Step 3: Use the λ value with the Parametric equation to find R:

$$x = \lambda + 1 = -1 \qquad y = -\lambda + 2 = 0 \qquad z = 2\lambda + 1 = -3 \qquad R(-1, 0, -3)$$

Step 4: Use the distance formula: $\overrightarrow{QR} = \boldsymbol{r} - \boldsymbol{q} = \begin{pmatrix} \mathbf{-1} \\ \mathbf{0} \\ \mathbf{-3} \end{pmatrix} - \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{-2} \\ \mathbf{-2} \\ \mathbf{-4} \end{pmatrix}$

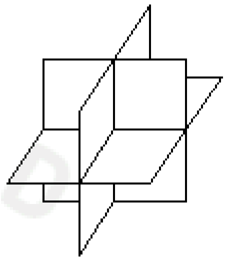
$$\left|\overrightarrow{\boldsymbol{QR}}\right| = \sqrt{(\mathbf{-2})^2 + (\mathbf{-2})^2 + (\mathbf{-4})^2} = \sqrt{\mathbf{24}} = \mathbf{2}\sqrt{\mathbf{6}}$$

Bk3 P73 Ex13
Q1, 3, 4

Intersection of Three Planes

There are SIX possible outcomes when three planes intersect:

1. **A point of intersection**



❷❷ Find the point of intersection of the planes:

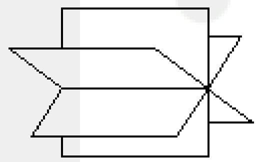
$$\pi_1:\ x - 2y + z = 8$$
$$\pi_2:\ 3x + y - z = 1$$
$$\pi_3:\ 2x - 2y + 3z = 18$$

Set up a matrix and solve: $\begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & -1 \\ 2 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ 18 \end{pmatrix}$

In augmented form [MATRIX]*:* $\left(\begin{array}{ccc|c} 1 & -2 & 1 & 8 \\ 3 & 1 & -1 & 1 \\ 2 & -2 & 3 & 18 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right) \Rightarrow (2, -1, 4)$

NB – look at marks awarded and judge if you can get away with using the graphics calculator!!

2. **A line of intersection**



❷❸ Find the line of intersection of the planes:

$$\pi_1:\ x + 2y - 2z = -7$$
$$\pi_2:\ x - 2y + z = 6$$
$$\pi_3:\ 3x + 2y - 3z = -8$$

Augmented (rref) matrix gives: $\left(\begin{array}{ccc|c} 1 & 2 & -2 & -7 \\ 1 & -2 & 1 & 6 \\ 3 & 2 & -3 & -8 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & -0.5 & -0.5 \\ 0 & 1 & -0.75 & -3.25 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Since we get REDUNDANCY, we let $z = \lambda$ *and multiply rows to get whole numbers:* $\left(\begin{array}{ccc|c} 2 & 0 & -1 & -1 \\ 0 & 4 & -3 & -13 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{pmatrix} 2x - z = -1 \\ 4y - 3z = -13 \\ z = \lambda \end{pmatrix}$

$$4y - 3z = -13 \Rightarrow 4y = 3\lambda - 13 \Rightarrow y = \frac{3\lambda - 13}{4}$$

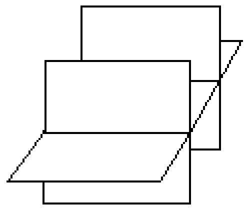
$$2x - z = -1 \Rightarrow 2x = \lambda - 1 \Rightarrow x = \frac{\lambda - 1}{2}$$

Parametric form: $x = \frac{\lambda - 1}{2},\ y = \frac{3\lambda - 13}{4},\ z = \lambda$

Cartesian form: $\frac{2x+1}{1} = \frac{4y+13}{3} = \frac{z}{1}$

3. **<u>Two lines of intersection</u>**

❷❹ Find the lines of intersection of the planes:



$$\pi_1:\ x - y + z = 10$$
$$\pi_2:\ 2x - y + 3z = 5$$
$$\pi_3:\ 4x - 2y + 6z = 7$$

Augmented (rref) matrix gives: $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 2 & -1 & 3 & 5 \\ 4 & -2 & 6 & 7 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

Since we get INCONSISTENCY, we have to check if there is a line of intersection between any pair of planes - see example ❷⓿:

- $\pi_1:\ x - y + z = 10$ *and* $\pi_2:\ 2x - y + 3z = 5$

$$\boldsymbol{n}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \boldsymbol{n}_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}\ \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{matrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & 1 \\ 2 & -1 & 3 \end{matrix} = \boldsymbol{-2i - j + k}$$

Using $z = 0$:

$$z = 0 \Rightarrow x - y = 10 \ \ \textit{and} \ \ 2x - y = 5$$

$$\Rightarrow \begin{matrix} x - y = 10 \\ 2x - y = 5 \end{matrix} \Rightarrow x = -5 \Rightarrow y = -15$$

$$(-5, -15, 0) \qquad \Rightarrow \frac{x+5}{-2} = \frac{y+15}{-1} = \frac{z}{1}$$

- $\pi_1:\ x - y + z = 10$ *and* $\pi_3:\ 4x - 2y + 6z = 7$

$$\boldsymbol{n}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \boldsymbol{n}_3 = \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}\ \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{matrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & 1 \\ 4 & -2 & 6 \end{matrix} = \boldsymbol{-4i - 2j + 2k = -2 : -1 : 1}$$

Using $z = 0$:

$$z = 0 \Rightarrow x - y = 10 \ \ \textit{and} \ \ 4x - 2y = 7$$

$$\Rightarrow \begin{matrix} 2x - 2y = 20 \\ 4x - 2y = 7 \end{matrix} \Rightarrow 2x = -13 \Rightarrow x = -\frac{13}{2} \Rightarrow y = -\frac{33}{2}$$

$$\left(-\frac{13}{2}, -\frac{33}{2}, 0\right) \qquad \Rightarrow \frac{x + \frac{13}{2}}{-2} = \frac{y + \frac{33}{2}}{-1} = \frac{z}{1}$$

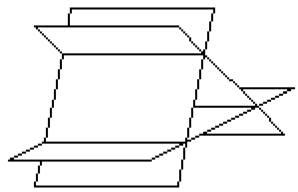
Multiply through by $\frac{2}{2}$ *:* $\frac{2x+13}{-4} = \frac{2y+7}{-2} = \frac{2z}{2}$

And simplify: $\frac{2x+13}{-4} = \frac{2y+7}{-2} = \frac{z}{1}$

NB - the normals for π_2 and π_3 are multiples of each other so they are parallel.

4. <u>**Three lines of intersection**</u>

❷❺ Find the lines of intersection of the planes:



$$\pi_1:\ x + 2y - 2z = -7$$
$$\pi_2:\ 3x + 2y - 3z = -15$$
$$\pi_3:\ 5x + 2y - 4z = -9$$

Augmented (rref) matrix gives: $\left(\begin{array}{ccc|c} 1 & 2 & -2 & -7 \\ 3 & 2 & -3 & -15 \\ 5 & 2 & -4 & -9 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & -0.5 & 0 \\ 0 & 1 & -0.75 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

Since we get INCONSISTENCY again, we have to check if there are lines of intersection between each pair of planes:

- $\pi_1:\ x + 2y - 2z = -7$ *and* $\pi_2:\ 3x + 2y - 3z = -15$

$$\boldsymbol{n}_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \boldsymbol{n}_2 = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}\ \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{matrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & -2 \\ 3 & 2 & -3 \end{matrix} = -2\boldsymbol{i} - 3\boldsymbol{j} - 4\boldsymbol{k} = \mathbf{2:3:4}$$

Using $z = 0$: $\quad z = 0 \Rightarrow x + 2y = -7$ *and* $3x + 2y = -15$

$$\Rightarrow \begin{matrix} x + 2y = -7 \\ 3x + 2y = -15 \end{matrix} \Rightarrow 2x = -8 \Rightarrow x = -4 \Rightarrow\ y = -\frac{3}{2}$$

$$\left(-4, -\frac{3}{2}, 0\right) \qquad \Rightarrow \frac{x+4}{2} = \frac{y + \frac{3}{2}}{3} = \frac{z}{4}$$

Multiply through by $\frac{2}{2}$ *:* $\quad \frac{2x+8}{4} = \frac{2y+3}{6} = \frac{2z}{8}$

And simplify: $\quad \frac{x+4}{2} = \frac{2y+3}{6} = \frac{z}{4}$

- $\pi_1:\ x + 2y - 2z = -7$ *and* $\pi_3:\ 5x + 2y - 4z = -9$

$$\boldsymbol{n}_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \boldsymbol{n}_3 = \begin{pmatrix} 5 \\ 2 \\ -4 \end{pmatrix}\ \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{matrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & -2 \\ 5 & 2 & -4 \end{matrix} = -4\boldsymbol{i} - 6\boldsymbol{j} - 8\boldsymbol{k} = \mathbf{2:3:4}$$

Using $z = 0$: $\quad z = 0 \Rightarrow x + 2y = -7$ *and* $5x + 2y = -9$

$$\Rightarrow \begin{matrix} x + 2y = -7 \\ 5x + 2y = -9 \end{matrix} \Rightarrow 4x = -2 \Rightarrow x = -\frac{1}{2} \Rightarrow\ y = -\frac{13}{4}$$

$$\left(-\frac{1}{2}, -\frac{13}{4}, 0\right) \qquad \Rightarrow \frac{x + \frac{1}{2}}{2} = \frac{y + \frac{13}{4}}{3} = \frac{z}{4}$$

Multiply through by $\frac{2}{2}$ *:* $\quad \frac{2x+1}{4} = \frac{2y+13}{6} = \frac{2z}{8}$

And simplify: $\quad \frac{2x+1}{4} = \frac{2y+13}{6} = \frac{z}{4}$

- $\pi_2:\ 3x+2y-3z=-15$ *and* $\pi_3:\ 5x+2y-4z=-9$

$$\boldsymbol{n}_1=\begin{pmatrix}3\\2\\-3\end{pmatrix},\ \boldsymbol{n}_3=\begin{pmatrix}5\\2\\-4\end{pmatrix}\quad \boldsymbol{n}_1\times\boldsymbol{n}_2=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\1 & 2 & -2\\3 & 2 & -3\end{vmatrix}=\boldsymbol{-2i-3j-4k=2:3:4}$$

Using $z=0$: $\quad z=0\Rightarrow 3x+2y=-15$ *and* $\quad 5x+2y=-9$

$$\Rightarrow \begin{matrix}3x+2y=-15\\5x+2y=-9\end{matrix}\Rightarrow 2x=6\Rightarrow x=3\Rightarrow\ y=-12$$

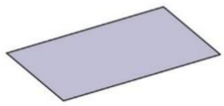
$$(3,-12,0)\qquad \Rightarrow\frac{x-3}{2}=\frac{y+12}{3}=\frac{z}{4}$$

Note that all 3 lines of intersection have the same direction vector i.e. they are parallel.

5. **A plane of intersection**

❷❻ Find the plane of intersection of the planes:

$$\pi_1:\ 2x-y+3z=4$$
$$\pi_2:\ 6x-3y+9z=12$$
$$\pi_3:\ 8x-4y+12z=16$$



Augmented (rref) matrix gives: $\left(\begin{array}{ccc|c}2 & -1 & 3 & 4\\6 & -3 & 9 & 12\\8 & -4 & 12 & 16\end{array}\right)=\left(\begin{array}{ccc|c}1 & -0.5 & 1.5 & 2\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right)$

Since we get DOUBLE REDUNDANCY then the planes coincide.

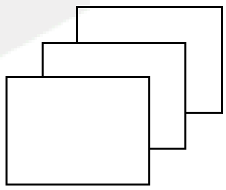
(<u>π_2 and π_3 are multiples of π_1</u>)

Plane of intersection is $2x-y+3z=4$ *- simplest form*

6. **No intersection**

❷❼ Show that these planes do not intersect:

$$\pi_1:\ 4x-8y+12z=12$$
$$\pi_2:\ 2x-4y+6z=2$$
$$\pi_3:\ 3x-6y+9z=6$$



Bk3 P78 Ex15 All

Augmented (rref) matrix gives: $\left(\begin{array}{ccc|c}4 & -8 & 12 & 12\\2 & -4 & 6 & 2\\3 & -6 & 9 & 6\end{array}\right)=\left(\begin{array}{ccc|c}1 & -2 & 3 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{array}\right)$

Since we get REDUNDANCY and INCONSISTENCY or if it had been DOUBLE INCONSISTENCY then the planes don't intersect (<u>π_2 and π_3 have normals that are multiples of π_1 so parallel</u>)

Vector Equation of a Plane

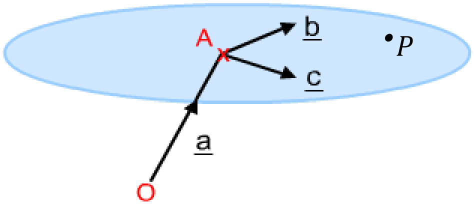
We've already met with the idea of finding the vector equation of a line in 3D:

$$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{u}$$

where R is any point on the line with position vector $\boldsymbol{r}$, $\boldsymbol{u}$ is the direction vector of the line and λ is a scalar. We can apply a similar method to defining the equation of a plane using this format.

Instead of one direction vector however, we would require two and these vectors must not be parallel to each other.

Any point P on the plane can be found using A as a starting point followed by a number of moves or combination of moves in the direction of $\underline{b}$ and/or $\underline{c}$. This gives us:



$$\underline{\boldsymbol{p}} = \boldsymbol{a} + t\underline{\boldsymbol{b}} + u\underline{\boldsymbol{c}}$$

This is the VECTOR EQUATION of the plane

Parametric Equation of a Plane

Using $\underline{\boldsymbol{p}} = \boldsymbol{a} + t\underline{\boldsymbol{b}} + u\underline{\boldsymbol{c}}$ with $A(a_1, a_2, a_3)$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ and

$P(x, y, z)$ then:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + u\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Thus $x = a_1 + tb_1 + uc_1$ $\quad y = a_2 + tb_2 + uc_2$ $\quad z = a_3 + tb_3 + uc_3$

This is known as the PARAMETRIC EQUATION of the plane

❷❽ Find the vector equation of the plane that contains the points A(1, 2, -1), B(-2, 3, 2) and C(4, 5, 2).

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

$$\underline{\boldsymbol{p}} = \boldsymbol{a} + t\underline{\boldsymbol{b}} + u\underline{\boldsymbol{c}} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} + u\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

❷❾ Find the equation of the plane in parametric form which is parallel to $\boldsymbol{i} + 2\boldsymbol{k}$ as well as $3\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$ and passes through the point $A(1, -2, 1)$

$$\underline{\boldsymbol{p}} = \boldsymbol{a} + t\underline{\boldsymbol{b}} + u\underline{\boldsymbol{c}} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + u\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$$

$x = 1 + t + 3u$ $\quad y = -2 - u$ $\quad z = 1 + 2t + 4u$

❸⓿ Find the Cartesian equation of the plane whose vector equation is:

$$\underline{\boldsymbol{p}} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + u\begin{pmatrix} 3 \\ -3 \\ -7 \end{pmatrix}$$

Reminder: The general equation of a plane is $ax + by + cz = k$ where $\boldsymbol{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normal to the plane and $k = \boldsymbol{n}.\boldsymbol{p}$ where P is a point on the plane.

Vector normal will be: $\boldsymbol{n} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -3 \\ -7 \end{pmatrix} = \begin{matrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & -1 \\ 3 & -3 & -7 \end{matrix} = \begin{pmatrix} -10 \\ 4 \\ -6 \end{pmatrix}$

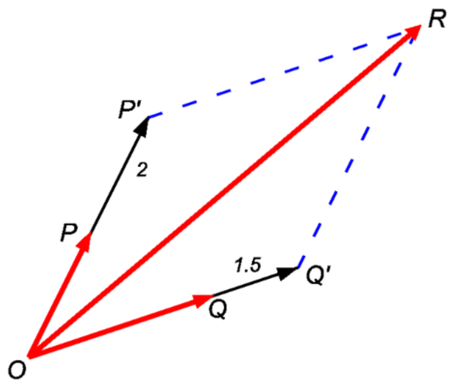
Dot product of normal and point on the line: $\begin{pmatrix} -10 \\ 4 \\ 6 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = -34$

Equation will be: $-10x + 4y - 6z = -34$

$\Rightarrow 5x - 2y + 3z = 17$

Bk3 P63 Ex8
Q1a, 2, 4a

Coplanar Vectors



Given the 3 vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{OR}$

If $\overrightarrow{OR} = t\overrightarrow{OP} + u\overrightarrow{OQ}$ then the vectors are said to be coplanar.

In the example shown:

$$\overrightarrow{OR} = 2\overrightarrow{OP} + \frac{3}{2}\overrightarrow{OQ}$$

and forms a parallelogram.

Note that $2\overrightarrow{OP}$ and $\frac{3}{2}\overrightarrow{OQ}$ define a plane in which $\overrightarrow{OR}$ lies.